ALGEBRA PRELIM

WINTER 2000

1. a. Show that a group G of order 56 is not simple.
   b. Show that a group G of order 72 is not simple.

2. Prove Cauchy's Theorem: If G is a finite group and p a prime divisor of G, then G contains an element of order p.
   Hint: First do the Abelian case, then use the class equation.

3. Let G be a group. Let $Z(G)$ be its center, $Aut(G)$ its automorphism group, and $Inn(G)$ be its inner automorphism group. An inner automorphism is one that is induced by conjugation by an element of G
   a. Show $Inn(G)$ is a normal subgroup of $Aut(G)$.
   b. Show $G/Z(G) \cong Inn(G)$.
   c. Show that if $Inn(G)$ is cyclic, then G is Abelian.

4. Let G be a group and H a subgroup of finite index. Show that there exists a *normal* subgroup N of G contained in H and also of finite index.
   Hint: If $[G : H] = n$, find a homomorphism of G into S_n whose kernel is contained in H.

5. a. Let F be a field and $a, b \in F$. Show that if $a^m = b^m$ and $a^n = b^n$, for m and n relatively prime positive integers, then $a = b$.
   b. Prove the same statement when the field F is replaced by a (commutative) integral domain D.

6. Let F be a field and $f(x) \in F[x]$ be a monic polynomial of positive degree.
   a. Show that there exists an extension field E of F ($F \subseteq E$) such that E contains a root of $f(x)$.
   b. Show that $f(x)$ has a splitting field K over F.

7. Let $\mathbb{Z}[i]$ be the ring of Gaussian integers. That is, $\mathbb{Z}[i] = \{a+bi \mid a, b \in \mathbb{Z} \text{ and } i^2+1 = 0\}$. In the ring $\mathbb{Z}[i]$ let A be the principal ideal generated by $1 + 3i$. Prove that $\mathbb{Z}[i]/A$ is isomorphic to Z_{10}, the ring of integers modulo 10.

8. Let K be the splitting field of $x^4 + 1$ over $\mathbb{Q}$. Show that K is a simple extension $K = \mathbb{Q}(\alpha)$, and find the Galois group of K over $\mathbb{Q}$. (You should be able to identify the Galois group up to isomorphism with a well-known group.)

9. Consider the quotient ring $F[x]/(x^2+1)$ for each of the following fields $F = \mathbb{Z}_2$, $F = \mathbb{Z}_3$, $F = \mathbb{Q}$, and $F = \mathbb{C}$, and determine in which of the four cases the quotient ring is a field.

Algebra Prelim Winter 2000

10. For each statement indicate TRUE or FALSE with brief justification:

a. If G is the group of invertible 2×2-matrices with entries in $\mathbb{F}_q$ the field of q elements, then G has order $(q^2 - 1)(q^2 - q)$.

b. If H and K are normal subgroups of a group G and $G/H \cong G/K$, then $H \cong K$.

c. If p is prime, then the group of units in the ring $\mathbb{Z}_{p^n}$ of integers modulo p^n is cyclic.

d. An irreducible polynomial $f(x)$ over a field K has no repeated roots in any extension field of K.

e. Every unique factorization domain is a principal ideal domain.