Algebra Preliminary Exam

March 7, 2002

1. Let $\mathcal{H}$ be the class of finite groups in which every subgroup is normal. (Such groups are called Hamiltonian groups.)

(a) Show that if $G \in \mathcal{H}$ and H is a subgroup of G, then $G/H \in \mathcal{H}$ and $H \in \mathcal{H}$.

(b) Let G be a finite group, and G^* a minimal normal subgroup of G such that $G/G^* \in \mathcal{H}$. Show that for any subgroup H of G which contains G^* (i.e., $G^* \leq H \leq G$) , H is normal in G and $G/H \in \mathcal{H}$.

2. Let G be a finite group.

(a) Show that if H is a proper subgroup of G then G is not the union of the conjugates of H. (Hint: Use a counting argument. How many distinct conjugates can H have?)

(b) Show that if G acts transitively on a set X of size at least 2, then some element $g \in G$ acts without fixed points (i.e., g is not contained in the stabilizer of any point). (Hint: Use part (a).)

3. Let F be a field, and let $A \in M_{n\times n}(F)$ be given. Set $C = \{B \in M_{n\times n}(F) | BA = AB\}$.

(a) Show that $F[A] = \{p(A) | p(x) \in F[x]\}$ is a subring of $M_{n\times n}(F)$ which is a vector space over F.

(b) Show that C is a subring of $M_{n\times n}(F)$ which is a vector space over F, and that $F[A] \subset C$.

(c) Compute C if $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and if $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. What is the dimension of $F[A]$ in each case?

4.(a) If R is a ring and M a left R-module, define what it means for M to be a free R-module.

(b) Show that every R-module is a homomorphic image of a free R-module.

(c) Let I be a non-zero ideal in a commutative ring R with identity. Show that I is a free R-module if and only if I is a principal ideal, generated by $\alpha \in R$ which is not a zero divisor in R.

5. Show that any two fields of the same finite order are isomorphic.

6. Find the Galois group of $(x^3 - 2)(x^3 - 3)$ over $\mathbb{Q}(\sqrt{3}i)$ and work out the Galois correspondence.