

Algebra Preliminary Exam
March 3, 2006

1. Let G be a group.
 - (a) Suppose that A and B are normal subgroups of G so that $G = AB$ and $A \cap B = \{e\}$. Show that $G \simeq A \times B$.
 - (b) Show that if $|G| = p^2$ for some prime p then G is abelian.
 - (c) Classify all groups of order 1225
2. A subgroup H of a group G is called *characteristic* in G if every automorphism of G maps H to itself. The *commutator subgroup* G' of a group G is the group generated by all commutators in G , i. e., all elements of the form $a^{-1}b^{-1}ab$ with $a, b \in G$. If H is a subgroup of a group G then the *Focal subgroup* of H with respect to G is $H^* = \{h^{-1}h^g \mid h \in H, h^g \in H, g \in G\}$. Here $h^g = g^{-1}hg$. Prove the following statements.
 - (a) Every characteristic subgroup is a normal subgroup.
 - (b) For any group G , $G' \triangleleft G$.
 - (c) If H is a subgroup of G then H' is a subgroup of H^* .
 - (d) $H^* \triangleleft H$ and H/H^* is abelian.
 - (e) If K is a subgroup of G such that $H^* \subseteq K \subseteq H$ then $K \triangleleft H$ and H/K is abelian.
3. Let R be a *PID*.
 - (a) Show that every nonzero prime ideal of R is maximal.
 - (b) If S is an integral domain and $\phi : R \rightarrow S$ a surjective ring homomorphism, show that either ϕ is an isomorphism or S is a field.
 - (c) Show that if $R[x]$ is a *PID*, then R is a field.
4.
 - (a) Let n be a positive integer. Show that $\mathbb{Z}/n\mathbb{Z}$ is a field if and only if n is prime.
 - (b) Construct the 4-element field \mathbb{F}_4 by exhibiting its addition and multiplication tables.
5. Let M and N be modules over a commutative ring R .
 - (a) Show that $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} = 0$ if $\gcd(m, n) = 1$.
 - (b) Characterize $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ in general.
 - (c) If G is a finite abelian group and $G \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} = 0$ for all primes p , show that $G = \{0\}$. Does the result remain true if G is infinite?
6. If M is a finitely generated module over a field K , prove that every submodule of M is free, and that every quotient of M is free.
7. Let E be the splitting field of $x^6 - 17$ over \mathbb{Q} .
 - (a) Find the Galois group G of E over \mathbb{Q} and identify it with a known group.
 - (b) Find a subfield F of E such that E has dimension 2 over F , and F is Galois over \mathbb{Q} . (Hint: Consider the center of G .)