WMU Algebra Prelim Exam
Spring 2008

Please complete the following 7 problems. This exam is closed book, closed notes. You will have 6 hours for this exam.

(1) Suppose K is a field of characteristic $p > 0$. Define the *Frobenius map* by $\phi(x) = x^p$ for all $x \in K$.
   (a) Prove that the Frobenius map is a homomorphism of K to itself.
   (b) Prove that the Frobenius map is an isomorphism if K is finite.
   (c) Give an example of a field K of characteristic p for which the Frobenius map is not an isomorphism.
   (d) Suppose r is a non-negative integer, $q = p^r$, K is the field with q elements, and F is the field with p elements. Prove that the Galois group $G(K/F)$ is cyclic of order r and generated by the Frobenius map.

(2) Let R be a commutative ring, and J its *Jacobson radical*, where by definition J is the intersection of all maximal ideals of R.
   (a) Prove that $x \in J$ iff $1 - rx$ is a unit for all $r \in R$.
   (b) (Nakayama's Lemma) Show that if M is any finitely generated R-module and $JM = M$, then $M = 0$. (HINT: Suppose $M \neq 0$ and let $m_1, \dots m_n$ be a minimal set of generators for M. Obtain a contradiction from expressing m_1 in terms of these generators.)

(3) If G acts transitively on a set X by $x \mapsto g.x$ for any $g \in G, x \in X$, show that
   (a) $G_x = \{g \in G \,|\, g.x = x\}$ is a subgroup of G.
   (b) $\{G_x \,|\, x \in G\}$ is a conjugacy class of subgroups of G.
   (c) $|X|$ divides $|G|$.

(4) $(F, +, *)$ is a *near field* if $(F, +)$ is an abelian group and $(F\backslash 0, *)$ is a group, and $a*(b+c) = a*b+a*c$. Show that $K = \{a \in F \,|\, (b+c)*a = b*a+c*a \quad \forall\, b, c \in F\}$ is a division subalgebra of F.

(5) Let G be a group of order $5075 = 52 \cdot 7 \cdot 29$. Let P be a Sylow 5-subgroup of G, let Q be a Sylow 7-subgroup of G, and let R be a Sylow 29-subgroup of G.
   (a) Show that P is a normal subgroup of G.
   (b) Show that G has a normal subgroup H of order $52 \cdot 29$.
   (c) Show that R is a normal subgroup of G.
   (d) Show that G has a subgroup K of order $52 \cdot 7$.
   (e) Show that K is normal if and only if G is Abelian.

(6) What are the Galois groups of $x^8 - 1$ and $x^8 + 1$ over $\mathbb{Q}$?

(7) Let $\mathbb{C}[x, y, z, w]$ be the polynomial ring with indeterminates x, y, z, and w. Let

$$I = \{f \in \mathbb{C}[x, y, z, w] \mid f(1, 2, 3, 4) = 0\}.$$

Prove that I is a maximal ideal of $\mathbb{C}[x, y, z, w]$.