Western Michigan University
Abstract Algebra Preliminary Exam
March 5, 2010

Instructions. Please write clearly, completely, and legibly. **Please put EACH problem on a separate piece of paper, with your name written at the top.** You have 6 hours to work this exam, starting at 9:00 a.m. and finishing at 3:00 p.m. Submit your solutions and this copy of your exam to the proctor.

(1) If G is a group, $Z(G)$ is the center of G, and $G/Z(G)$ is cyclic, then G is abelian.

(2) Let R be a commutative ring with identity.
(a) Define what we mean by a prime ideal in R.
(b) Is the intersection of two prime ideals necessarily a prime ideal?
(c) If
$$P_1 \supset P_2 \supset \cdots \supset P_n \supset \cdots$$
is a decreasing sequence of prime ideals in R, show that
$$\bigcap_{n=1}^{\infty} P_n$$
is a prime ideal in R.
(d) Show that every commutative ring R has a minimal prime ideal, i.ė., a prime ideal P which does not properly contain any other prime ideal.

(3) Find the Galois group of $x^3 - 10$ over $\mathbb{Q}$ and over $\mathbb{Q}(\sqrt{-3})$.

(4) Suppose $T : V \to W$ is a linear transformation of k-vector spaces for a field k. Recall there is a corresponding linear transformation of dual spaces $T^* : W^* \to V^*$ defined by
$$(T^*(f))(v) = (f \circ T)(v) \quad \forall v \in V, f \in W^*.$$
(Remember, $W^* = \mathrm{Hom}_k(W, k)$ is the space of linear transformations of W to k, and likewise for V^*.)
(a) First, prove that T^* is indeed linear.
(b) If $T' : W \to Z$ is another linear transformation of k-vector spaces, prove that $(T' \circ T)^* = T^* \circ (T')^*$.
(c) Now, suppose $\dim(V) = n$ and $\dim(W) = m$. Then, prove that for the fixed choice of bases $\mathcal{W} = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ of W and $\mathcal{V} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of V, the matrix of the linear transformation T^* with respect to the dual bases $\mathcal{W}^*$, $\mathcal{V}^*$ is A^T, where A is the matrix $[T]_{\mathcal{V}\mathcal{W}}$ representing T with respect to the bases $\mathcal{V}$ and $\mathcal{W}$.

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(5) Classify, up to isomorphism, all groups of order 12. Where do the alternating group A_4 and the dihedral group D_6 fall in your classification?

(6) Let M be a monoid. (Recall that a **monoid** is a set closed under an associative binary operation, with an identity element.) Show that:
  (a) $F := \{f \mid f : M \to M\}$ is a monoid under composition of functions.
  (b) M is isomorphic to a submonoid of F.

(7) Let $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ be an exact sequence of left modules over a ring R. Show that the following are equivalent:
  (a) there exists a homomorphism $\phi : M'' \to M$ such that $g \circ \phi = \mathrm{id}_{M''}$;
  (b) there exists a homomorphism $\psi : M \to M'$ such that $\psi \circ f = \mathrm{id}_{M'}$;
  (c) there exists a submodule $N \subset M$ such that $M = N \oplus \mathrm{im} f$.

(8) Consider the general linear group $GL_n(k)$ for k an algebraically closed field and $n \geq 1$. Let $M_n(k)$ be the k-vector space of all $n \times n$ matrices with entries from k.
  (a) Prove that $GL_n(k)$ acts on $M_n(k)$ by conjugation.
  (b) Under the conjugation action, a $GL_n(k)$-orbit $\mathcal{O}$ in $M_n(k)$ is called a **nilpotent orbit** if there is a representative $X \in \mathcal{O}$ for which X is a nilpotent matrix.
    (i) Show that if $X \in \mathcal{O}$ is nilpotent, then $Y \in \mathcal{O}$ is nilpotent for every $Y \in \mathcal{O}$.
    (ii) Let $\mathcal{N}_n(k)$ be the set of all nilpotent orbits. Show that $\mathcal{N}_n(k)$ is finite, and for the particular case when $n = 5$, give a concrete representative of each element of $\mathcal{N}_n(k)$.