WMU Algebra Preliminary Exam

March 9, 2012

This exam is closed book, closed notes. There are 7 problems. You have six hours for this exam. Please put each problem on a separate sheet of paper, with your name at the top.

1. Determine, up to isomorphism, all groups of order 203.
2. Let G be a finite group with H and K subgroups such that $G = HK$. Show that
$$G/(H \cap K) \simeq G/H \times G/K$$
as G-sets. (First define what "isomorphic as G-sets" means.) Note: It is not assumed that either H or K is normal.
3. Show that any Euclidean domain R is a PID.
4. Let R be the ring $(\mathbb{Z}/2\mathbb{Z})[x]/(x^3 + x)$.
   (a) Classify all ideals of R.
   (b) Find all the units of R.
   (c) Find the idempotent elements of R (i.e., those that satisfy $e^2 = e$).
5. Recall that if R is a ring, then an R-module is *torsion* if for every $m \in M$ there is a nonzero ring element r such that $rm = 0$. Let R be an integral domain.
   (a) Show that every finitely generated torsion R-module has a nonzero annihilator. (Recall $Ann(M) = \{r \in R : rM = 0\}$.)
   (b) Give an example of a torsion R-module whose annihilator is the zero ideal. (Remember that you must show everything.)
6. Let $K = \mathbb{Q}(\sqrt{2} + \sqrt{3}, \sqrt{5})$.
   (a) Determine $[K : \mathbb{Q}]$.
   (b) Compute $Gal(K/\mathbb{Q})$.
7. Let $B = B(-,-) : V \times V \longrightarrow F$ be a symmetric bilinear form on a finite dimensional vector space V over a field F. For a subspace $W \subset V$, we define the annihilator subspace of W in V:
$$W^{\perp} = \{x \in V : B(x, w) = 0 \text{ for every } w \in W\}.$$
Assume further that B is nondegenerate, that is $V^{\perp} = 0$. Show that:
   (a) The map $v \mapsto B(v, -)$ defines an isomorphism between V and V^*;
   (b) Every $\lambda \in W^*$ extends to an element of V^*;
   (c) There is a natural isomorphism of vector spaces from $V/W^{\perp}$ to W^*;
   (d) $\dim(W^{\perp}) = \dim(V) - \dim(W)$;
   (e) $(W^{\perp})^{\perp} = W$.