WESTERN MICHIGAN UNIVERSITY
ABSTRACT ALGEBRA COMPREHENSIVE EXAMINATION
AUGUST 26, 2013

Instructions: There are seven questions, some with several parts. Write your solution to each problem on a separate sheet of paper, with your name at the top.

(1) For each n, let S_n be the symmetric group on n letters, and let G be a finite group. Prove the following:
  (a) If the order $|G|$ of G is a product of two distinct primes then G is not simple.
  (b) If $|G| = p^2q$ for p and q distinct primes, then G is not simple.
  (c) If $n \leq 4$, then S_n has no non-abelian simple subgroups.
  (d) If G is a non-abelian simple group and H is a proper subgroup of G, then $|G : H| \geq 5$.

(2) Let $V = \mathbb{R}^4$. Let S be the set of all two-dimensional subspaces of V, and fix $W \in S$. Let $G = GL(V)$ (the group of invertible linear operators on V) act naturally on S, and let $H = \{g \in G : g \cdot W = W\}$. Show that H has exactly three orbits on S.

(3) Let k be a field, and let R be the ring of $n \times n$ matrices with entries in k. Let $V = k^n$. Show that V is a simple R-module, with each element of R acting by the associated linear transformation.

(4) Let R be a commutative ring with identity, I an ideal of R, and M a module over R. Let

$$S = \{a \cdot m : a \in I, m \in M\}.$$

  (a) Is S an R-submodule of M? Prove or give a counterexample.
  (b) Let $I \cdot M$ be the R-submodule of M generated by S. Prove that $M \otimes_R (R/I) \cong M/(I \cdot M)$.

(5) Let R be commutative ring with identity. The **Krull dimension** of R, denoted by $\dim(R)$, is the supremum of the indices d of all strictly increasing chains

$$P_0 \subsetneqq P_1 \subsetneqq \cdots \subsetneqq P_d$$

of prime ideals in R. Define $\dim(R) = \infty$ if there is no finite supremum.
  (a) Determine the Krull dimension $\dim(R)$ if R is a field.
  (b) Determine the Krull dimension $\dim(R)$ if $R = \mathbb{Z}$.
  (c) Show that an ideal P in R is prime if and only if R/P is an integral domain.
  (d) Show that $\dim(R) \geq n$ if $R = k[x_1, \ldots, x_n]$ for k a field.

(6) Let $E = \mathbb{Q}(a)$ for $a = \sqrt{1 + \sqrt{2}}$.
  (a) Find $[E : \mathbb{Q}]$.
  (b) Identify $Gal(E/\mathbb{Q})$.
  (c) How many subfields of E are there?

(7) Consider the polynomial $f(x) = x^7 - 1 \in \mathbb{F}_2[x]$.
  (a) Find a splitting field K for $f(x)$.
  (b) Factor $f(x)$ into irreducible polynomials over $\mathbb{F}_2[x]$.
  (c) Show that the squaring map $\varphi(a) = a^2$ is an automorphism of K fixing $\mathbb{F}_2$, and find the orbits of φ.