chaps 1-12
(I [illegible] 2 + 7)

Graph Theory Preliminary Exam

26 June 1999

Instructions: Do **all** four problems in **Part A** and do **exactly** four of the five problems in **Part B**. Indicate clearly which problem in Part B you have omitted. Each problem in Part A is valued at 10 points, while each problem in Part B is valued at 15 points. Hand in **eight** problems only. Begin each problem on a new sheet of paper and only write on one side of the paper. You have six hours to complete the exam. When you are ready to hand in your exam, assemble your solutions in numerical order, write you name on the front page, and initial all other pages.

Part A: Do all four problems. (10 points for each problem)

A1 Prove that every nontrivial connected graph contains at least two vertices that are not cut-vertices.

A2 Let $G = K_{3,3} - e$, where $K_{3,3}$ has partite sets $V_1 = \{v_1, v_3, v_5\}$ and $V_2 = \{v_2, v_4, v_6\}$ and $e = v_5v_6$.

(a) Describe a 2-cell embedding of G on a surface S_k, where $k \neq \text{gen}(G)$, by giving an appropriate 6-tuple of cyclic permutations. What is k in your example? Include all details.

(b) On what surfaces is G 2-cell embeddable? Verify your answer.

A3 The *subdivision graph* $S(G)$ of a graph G is that graph obtained from G by replacing each edge uv of G by a new vertex w and the two edges uw and vw. Prove that if $S(G)$ is hamiltonian, then G is eulerian.

A4 Recall that $\omega(G)$ denotes the clique number of G (the maximum number of vertices in a complete subgraph of G). Certainly, $\chi(G) \geq \omega(G)$ for every graph G. Define a graph G to be *perfect* if $\chi(F) = \omega(F)$ for every induced subgraph F of G (including $F = G$).

(a) Explain why every complete graph and every bipartite graph is perfect.

(b) Prove that a critically k-chromatic graph G is perfect if and only if $G = K_k$.

Part B: Do exactly four of the five problems. (15 points for each problem)

B1 Let T_n denote the transitive tournament of order n. Hence T_n can be defined as that digraph with vertex set $\{v_1, v_2, \cdots, v_n\}$ such that (v_i, v_j) is an arc of T_n if and only $i < j$. Also, recall that an oriented graph G is obtained by assigning a direction to each edge of some graph G. For oriented graphs D_1 and D_2, define the *tournament Ramsey number* $TR(D_1, D_2)$ as the smallest positive integer p for which any 2-coloring of the arcs of T_p results in either a red D_1 or a blue D_2.

(a) Determine $TR(T_2, T_m)$.

(b) Find necessary and sufficient conditions on D_1 and D_2 in order to guarantee the existence of $TR(D_1, D_2)$.

(c) Let $\vec{P}_{n+1}$ denote the directed path of order $n+1$. Prove that

$$TR(\vec{P}_{n+1}, \vec{P}_{m+1}) = nm + 1.$$

[Hint: Let r_i denote the maximum order of a red directed path starting at vertex v_i and let b_i denote the maximum order of a blue directed path starting at v_i.]

B2 Let the graph G be obtained from $K_{2,6}$, whose partite sets are $V_1 = \{u_1, u_2\}$ and $V_2 = \{v_1, v_2, \cdots, v_6\}$, by adding two new vertices w_1, w_2 and the seven edges u_1u_2, w_1v_i $(1 \le i \le 3)$, and w_2v_j $(4 \le j \le 6)$. The graph G is shown below.

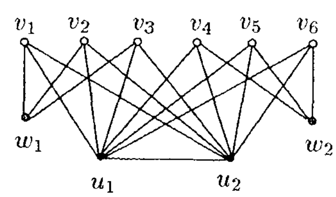

Figure 1: The graph G

(a) Use Tutte's characterization of graphs with 1-factors to show that G does not have a 1-factor.

(b) Petersen's theorem states that if G is a bridgeless, cubic graph, then G has a 1-factor. Show that Petersen's theorem can be extended somewhat by proving that if G is a bridgeless graph with degree set $\{3, 7\}$ such that G has exactly one vertex of degree 7, then G has a 1-factor.

(c) Show that the result in (b) cannot be extended any further by giving an example of a bridgeless graph G with degree set $\{3, 7\}$ such that G has two vertices of degree 7 and no 1-factor.