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Graph Theory Preliminary Examination
February 26, 2000

Part A: Do 4 out of the following 5 for 10 points each.

1. Let m be an even integer and let G be a graceful graph of size m. Show that K_{3m+1} can be decomposed into $3m+1$ copies of G and a copy of some m-regular graph H.

2. For a positive integer k, let G be a $(2k+2)$-regular graph of order $4k+1$.

   a. Show that G is hamiltonian.

   b. For any hamiltonian cycle C of G, let $H = G - E(C)$. Prove that H is of class 2.

3a. State Frucht's Theorem.

b. Illustrate the proof of Frucht's Theorem for the group $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_4$.

4a. State the theorem giving Euler's formula for plane graphs.

b. State and prove a generalization involving the parameter $k(G)$. (You do not need to prove the special case of (i) above.)

5. For any two vertices u, v of a strong digraph D, the strong distance between u and v, denoted $\text{sd}(u,v)$, is the size of a smallest strong subdigraph of D which contains both u and v. The strong eccentricity of u, denoted $\text{se}(u)$, is defined to be $\max_{v \in V(D)}\{\text{sd}(u,v)\}$. The strong radius of D, denoted $\text{srad}(D)$, is defined to be $\min_{u \in V(D)}\{\text{se}(u)\}$, while the strong diameter of D, denoted $\text{sdiam}(D)$, is defined to be $\max_{u \in V(D)}\{\text{se}(u)\}$. Prove that $\text{srad}(D) \leq \text{sdiam}(D) \leq 2\text{srad}(D)$.

Part B: Do 4 out of the following 5 for 15 points each.

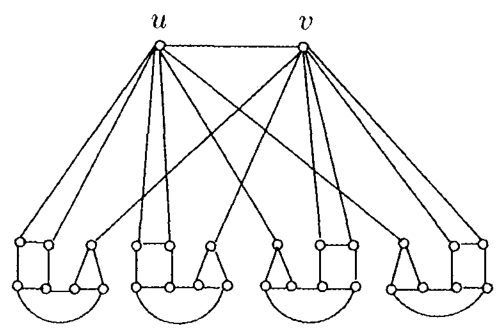
1a. Prove that if G is an r-regular graph of order n, then

$$\gamma(G) \geq \lceil n/(r+1) \rceil,$$

where $\gamma(G)$ is the domination number of G.

b. Show that the inequality in (a) can be strict by giving an example of a connected r-regular graph G of order n (for some r and n) such that $\gamma(G) > \lceil n/(r+1) \rceil$. Justify that your example has the domination number.

2a. Use Tutte's characterization of graphs with 1-factors to show that the graph G shown below does not have a 1-factor.



b. Petersen's theorem states that if G is a bridgeless, cubic graph, then G has a 1-factor. Show that Petersen's theorem can be extended somewhat by proving that if G is a bridgeless graph having exactly one vertex of degree 7 and all others of degree 3, then G has a 1-factor.

3a. Let T be any tree of order m and let G be a graph with minimum degree $m-1$. Prove that T is a subgraph of G.

b. Assume T is not a star $K_{1,m-1}$ and G is connected, G has minimum degree $m-2$ but $G \neq K_{m-1}$. Prove that T is a subgraph of G.

4a. Construct a cubic graph of order 12 with the maximum possible girth, and show that this truly is the maximum girth.

b. Do the same for order 14.

5 A rainbow graph has every edge a different color. The rainbow ramsey number $RR(G_1, G_2) = N$ is the smallest order N for which, whenever we color K_N, no matter how many or few colors we use, we must get either a monochromatic G_1 or a rainbow G_2. Prove that $RR(G_1, 3K_2)$ can be bounded in terms of the traditional ramsey number, namely

$$r(G_1, G_1) \leq RR(G_1, 3K_2) \leq r(G_1, G_2) + 4$$