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Graph Theory Preliminary Examination
May 25, 2001

Part A: Do 4 out of the following 5 for 10 points each.

1. Let S be a finite nonempty set and let $\{S_x\}_{x \in S}$ be a collection of subsets of S (thus there is one subset S_x for each element $x \in S$) having the following two properties:

   (i) $x \notin S_x$ for each $x \in S$;

   (ii) if $y \in S_z$, then $z \in S_y$ for all $y, z \in S$.

   Prove that there is an even number of subsets in the collection $\{S_x\}_{x \in S}$ having odd cardinality.

2. Let G_1 and G_2 be two k-connected graphs, where $k \geq 2$, and let $\mathcal{G}$ be the set of all graphs obtained by adding k edges between G_1 and G_2. Determine $\max\{\kappa(G) : G \in \mathcal{G}\}$, where $\kappa(G)$ is the connectivity of a graph G.

3. Let T be a tournament with the property that every vertex of T belongs to a directed 3-cycle. Let u and v be distinct vertices of T. Prove that if $|od\ u - od\ v| \leq 1$, then T contains both a directed $u - v$ path and a directed $v - u$ path.

4. Determine the Ramsey number $r(C_4, K_3)$.

5a. Define the n-cube Q_n, for $n \in \mathbb{N}$.

b. Show that $Q_4 = C_4 \times C_4$, the Cartesian product of two 4-cycles.

c. Show that $1 \leq \nu(Q_4) \leq 8$, where ν denotes the crossing number.

Part B: Do 4 out of the following 5 for 15 points each.

1. Let G be a graph with $V(G) = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and let $\{\pi_0, \pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6, \pi_7\}$ describe a 2-cell embedding of G on the surface S_k, where

$$\pi_0 = (1\ 3\ 2\ 7\ 5\ 6), \qquad \pi_4 = (5\ 7\ 6\ 3\ 1\ 2),$$
$$\pi_1 = (2\ 4\ 3\ 0\ 6\ 7), \qquad \pi_5 = (6\ 0\ 7\ 4\ 2\ 3),$$
$$\pi_2 = (3\ 5\ 4\ 1\ 7\ 0), \qquad \pi_6 = (7\ 1\ 0\ 5\ 3\ 4),$$
$$\pi_3 = (4\ 6\ 5\ 2\ 0\ 1), \qquad \pi_7 = (0\ 2\ 1\ 6\ 4\ 5)$$

   I. a. Show that G is a Cayley graph for the group $\Gamma = \mathbb{Z}_8$ and generating set $\Delta = \{1, 2, 3\}$.

   b. Find $\overline{G}$.

   c. Use $\overline{G}$ to describe G.

   d. Why is $\mathbb{Z}_8$ a subgroup of $\text{Aut}(G)$?

   e. Make a good guess for $|\text{Aut}(G)|$.

   II. f. Find k (where G embeds on S_k as described above) Hint: just calculate the two orbits beginning 0-1- and 1-0- and note that the cyclic nature of the rotational embedding scheme forces all other orbits to have the same length.

   g. Find $\text{gen}(G)$.

   h. Find $\text{gen}_M(G)$.

   i. Find all integers m so that G 2-cell embeds on S_m.

2. A *partial balanced incomplete block design* (V, B) of order n with block size k is an n-set V together with a set B of k-subsets (called *blocks*) of V such that every pair of elements of B occurs in at most one block of B. One can view such a partial design as a partition of a subset of the edges of K_n into copies of K_k.

   A partial balanced incomplete block design (V, B) is said to be *finitely embedded* in a balanced incomplete block design (V', B') of order v if $V \subseteq V'$ and $B \subseteq B'$. That is, (V', B') is a partition of the edge set of K_v into copies of K_k in which the copies of K_k in the original design are preserved.

   Show that every partial balanced incomplete block design can be finitely embedded.

3. Suppose that the edges of a graph G are properly colored with colors from the set $\{1, 2, \ldots, k\}$. Let Γ be a proper edge-coloring of G, and let c_i denote the number of edges of G colored i in the coloring Γ. We say that Γ is an *equalized* edge-coloring if $|c_i - c_j| \leq 1$, for all pairs i, j, where $1 \leq i, j \leq k$. Prove that if G admits a proper k-edge-coloring, the G admits an equalized, proper k-edge-coloring.

4. Let G be a graph with domination number $\gamma(G) \geq 2$ and let S be a minimal dominating set of G such that $<S>$ is connected. Prove that for each vertex $v \in S$, there exists a vertex $u \in V(G) - S$ such that u is adjacent to v but u is adjacent to no vertex in $S - \{v\}$.

5. Let G be a graph with no even cycles.

   a. Prove that no cycle in G can contain a chord.

   b. Determine $\max |E(G)|$ as a function of $n = |V(G)|$.