Graph Theory Preliminary Exam 2007

Instructions: Do **exactly** four of the five problems in **Part A** and do **exactly** four of the five problems in **Part B**. Indicate clearly which problem in Part A and which problem in Part B you have omitted. Each problem in Part A is valued at 10 points, while each problem in Part B is valued at 15 points. Hand in **eight** problems only. Begin each problem on a new sheet of paper and write on one side of the paper only. You have six hours to complete the exam. When you are ready to hand in your exam, assemble your solutions in numerical order, write you name on the front page, and initial all other pages.

Part A

A1 (10 points) For a nonempty forest F, the rainbow Ramsey number $RR(F)$ is the smallest positive integer n such that if each edge of the complete graph K_n is colored from any number of colors, then either a monochromatic F or a rainbow F is produced. Determine (with proof) $RR(K_{1,k})$ for each integer $k \geq 3$.

A2 (10 points) Let $s : d_1, d_2, \cdots, d_n$ be a non-increasing sequence of $n \geq 2$ positive integers, that is, $d_1 \geq d_2 \geq \cdots \geq d_n \geq 1$. Characterize those sequences s that are degree sequences of some tree of order n. That is, state and prove a theorem of the form:

The sequence s is the degree sequence of some tree of order n if and only if ....

A3 (10 points) Let T be a tournament of order $n \geq 6$ that is not strong, and let u and v belong to different, nontrivial, strong components. Let $s(u)$ and $s(v)$ be the scores of u and v, where (u, v) is an arc of T.

(a) Determine (with proof), as a function of n, the minimum value of $s(u) - s(v)$.

(b) Determine (with proof), as a function of n, the maximum value of $s(u) - s(v)$.

(c) Suppose that T is now a strong tournament. What is the smallest number of Hamiltonian cycles present in T as a function of n.

A4 (10 points) Let G be a locally finite infinite graph. That is, G has a countably infinite vertex set, and each vertex has finite degree. Let G be connected. Prove or disprove:

(a) If G has a finite set of vertices A such that $G - A$ has three or more infinite components, then G does not have a (two-way-infinite) Hamiltonian path.

(b) If G has an infinite set of vertices A such that $G - A$ has three or more infinite components, then G does not have a (two-way-infinite) Hamiltonian path.

A5 (10 points) Prove this theorem. If G is a graph of diameter 2, then $\kappa_1(G) = \delta(G)$.

Part B

B1 (15 points) Let G be a 4-regular graph of order 10 and size m.

(a) What can be deduced about the planarity of G by comparing the numbers m and $3n-6$?

(b) Prove or disprove: There is a planar 2-connected 4-regular graph of order 10.

(c) Show that the 2-connected 4-regular graph H of order 10 shown in Figure 1 does not contain K_5 as a subgraph but does contain K_5 as a minor. What can you conclude from this?

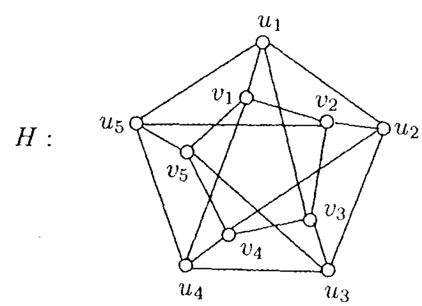

Figure 1: A 2-connected 4-regular graph of order 10

(d) Prove that if G is a graph of order $n \geq 5$ and size $m \geq 3n-5$, then G is need not contain K_5 as a subgraph but must contain a subgraph with minimum degree 4.

B2 (15 points) For two nonempty graphs F_1 and F_2, let $r(F_1, F_2)$ denote the Ramsey number of F_1 and F_2.

(a) Let F and H be two nonempty graphs, where $x \in V(F)$ and $y \in V(H)$. Suppose that F' is isomorphic to $F - x$ and H' is isomorphic to $H - y$. Prove that $r(F, H) \leq r(F', H) + r(F, H')$.

(b) Use (a) to give an upper bound for $r(K_3, K_4)$.

B3 (15 points)

(a) Show that the cubic graph $K_{3,3}$ has a nowhere-zero 3-flow but no nowhere-zero 2-flow.

(b) Show that the cubic graph $K_3 \times K_2$ has a nowhere-zero 4-flow but no nowhere-zero 3-flow.

(c) It is known that the Petersen graph is a nonplanar graph. Show, however, that the Petersen graph can be embedded on the torus.

(d) Use (c) to show that the Petersen graph has a nowhere-zero 5-flow.

(e) Prove that the Petersen graph does not have a nowhere-zero 4-flow.

B4 (15 points) Let G and its complement $\overline{G}$ both be connected graphs with order $n \geq 5$.

(a) Prove that if the diameter of G is at least 3, then the diameter of its complement is at most 3.

(b) What diameters are possible for self complementary graphs with at least 3 vertices?

(c) If the diameter of G is 2, what is the smallest and largest diameter of its complement $\overline{G}$, expressed in terms of n?

B5 (15 points) Denote the clique number of a graph G of order n by $\omega(G)$ and the independence number of G by $\beta(G)$.

(a) Let $k \geq 2$ be an integer. For each integer i with $1 \leq i \leq 2k+1$, let G_i be a copy of K_k. Then the graph G of order $2k^2 + k$ is obtained from the graphs $G_1, G_2, \ldots, G_{2k+1}, G_{2k+2} = G_1$ by joining each vertex in G_i to every vertex in G_{i+1}. Determine $\chi(G)$ with explanation and determine whether G is perfect.

(b) It is known that

$$\omega(G) \leq \chi(G) \leq n - \beta(G) + 1 \leq n.$$

Show that $\chi(G)$ can never be closer to n than to $\omega(G)$. In particular, use induction on the nonnegative number $|V(G)| - \omega(G)$ to prove that

$$\chi(G) \leq \frac{n + \omega(G)}{2}.$$

(c) It is also known that $\chi(G)$ can never be closer to $n - \beta(G) + 1$ than to $\omega(G)$; that is,

$$\chi(G) \leq \frac{\omega(G) + n - \beta(G) + 1}{2}. \qquad (1)$$

[Do not prove (1).] Use (1) to verify the Nordhaus-Gaddum Theorem: If G is a graph of order n, then $\chi(G) + \chi(\overline{G}) \leq n + 1$.