WMU Department of Mathematics
Algebra Comprehensive Exam
May 30, 2014

Instructions. Work all of these problems and write their solutions clearly and completely (and legibly). Write your solution to each problem on a separate sheet of paper, with your name at the top of each page. You have 6 hours to complete this exam.

1. Let R be a commutative ring with 1, and let I and J be ideals in R with $I + J = R$.
   (a) Let $a, b \in R$. Prove that there exists $c \in R$ such that $c \equiv a \bmod I$ and $c \equiv b \bmod J$.
   (b) Deduce from the above that $R/(I \cap J)$ is isomorphic to the direct product $R/I \times R/J$.
2. Show that any linear operator on a finite dimensional vector space (over a field of characteristic not equal 2) which satisfies $T^2 = I$ is diagonalizable.
3. Let G be a group, and let N be a normal subgroup of G. Let $\phi : G \to G/N$ be the canonical homomorphism. Let H be another group, and $f : G \to H$ be a homomorphism.
   (a) Show that there is a homomorphism $\bar{f} : G/N \to H$ such that $\bar{f} \circ \phi = f$ if and only if $N \leq Ker(f)$.
   (b) Assuming $\bar{f}$ exists, show that it is unique.
4. Consider the polynomial $f = x^8 - 1 \in \mathbb{F}_3[x]$.
   (a) Find a splitting field $\mathbb{K}$ for f.
   (b) Factor f into irreducible polynomials over $\mathbb{F}_3[x]$.
   (c) Show that the cubing map $\phi(a) = a^3$ is an automorphism of $\mathbb{K}$ fixing $\mathbb{F}_3$, and find the orbits of the group generated by ϕ.
5. Let G be a group of order $p^2 q$, where p and q are distinct primes. Show that G is not simple.
6. Suppose R is a ring with 1 and M is a left R-module. Let $N_1 \subseteq N_2 \subseteq \cdots$ be an ascending chain of submodules of M. Show that $\bigcup_{i=1}^{\infty} N_i$ is a submodule of M.