WMU Department of Mathematics
Algebra Comprehensive Exam
March 7, 2014

Instructions. Work all of these problems and write their solutions clearly and completely (and legibly). Write your solution to each problem on a separate sheet of paper, with your name at the top of each page. You have 6 hours to complete this exam.

1. Suppose G is a finite group of odd order. Show that if H is a normal subgroup of G of order 5, then H is contained in the center of G. (Hint: What is the automorphism group of H?)

2. Suppose A is an 8×8 matrix with complex entries such that

- $dim_{\mathbb{C}}(ker(A - 2I)) = 2$,
- $dim_{\mathbb{C}}(ker(A - 2I)^2) = 3$,
- $dim_{\mathbb{C}}(ker(A - 3I)) = 3$,
- $dim_{\mathbb{C}}(ker(A - 3I)^2) = 4$,
- $dim_{\mathbb{C}}(ker(A - 3I)^3) = 5$.

(a) What is the characteristic polynomial of A? (b) What is the minimal polynomial of A? (c) What is the Jordan Canonical Form of A? (d) What is the Rational Canonical Form of A?

3. Let G be a group, $S(G)$ the permutations of G, $R(G)$ the subgroup of $S(G)$ generated by right multiplications, $L(G)$ the subgroup of $S(G)$ generated by left multiplications, $A(G)$ the automorphisms of G, and $H(G)$ the normalizer of $R(G)$ in $S(G)$. Prove that (a) $R(G)$ is the centralizer of $L(G)$ in $S(G)$ and (b) the subgroup of $H(G)$ fixing the identity of G is $A(G)$.

4. Let $R = \mathbb{F}_2[X, Y]/(X^2, XY, Y^2)$, where $\mathbb{F}_2$ is the field of two elements. Find all the ideals I of R and their annihilators $\text{ann}(I) = \{r \in R : rx = 0,\ \text{all } x \in I\}$. Does the ring R have the property that $\text{ann}(\text{ann}(I)) = I$, for all ideals I? Justify all of your assertions.

5. Determine the Galois group of $x^4 - 4$ over the field $\mathbb{Q}$ of rational numbers. How many intermediate fields are there between $\mathbb{Q}$ and the splitting field of $x^4 - 4$? Identify them and sketch a diagram indicating inclusions among them.

6. Let F be a field. (a) Prove that the polynomial ring $F[x]$ is a Euclidean domain. (b) Prove that $F[x, y]$ is not a Euclidean domain.

7. Suppose R is a ring and

$$0 \longrightarrow A \xrightarrow{\iota} B \xrightarrow{\pi} C \longrightarrow 0$$

is a short exact sequence of R-modules. Prove that the sequence is split iff B is isomorphic to the direct sum $A \oplus C$. (To say that it is split means that there is an R-module homomorphism $\sigma : C \to B$ such that $\pi \circ \sigma = \mathrm{id}_C$.)