Graph Theory Preliminary Examination

December 15, 2015

Instructions

Do **exactly four** of the five problems in **Part A** and do **exactly four** of the six problems in **Part B**. Indicate clearly which problem in Part A and which two problems in Part B you have omitted. Each problem in Part A is valued at 10 points, while each problem in Part B is valued at 15 points.

Hand in **eight** problems only. Begin your solution of each problem on a new sheet of paper and write on one side of the paper only. You have six hours to complete the exam.

When you are ready to hand in your exam, assemble your solutions in numerical order and write your name on the front page.

Part A

A1 Prove that if the diameter of a connected graph G is at least 3, then the diameter of its complement $\overline{G}$ is at most 3.

A2 Prove that if G is a connected graph that is neither a tree nor a cycle, then G contains a vertex v such that $G - v$ is connected and is not a tree.

A3 Show that there is no 3-chromatic graph G such that $G - v$ is 3-critical for every vertex v of G.

A4 A graph G of order 10 and size 30 is 2-cell embedded on the torus.

(a) How many regions are there in this embedding?

(b) Can G be embedded in the plane?

(c) What is the genus $\gamma(G)$ of G?

(d) What is the genus $\gamma(G + uv)$ of $G + uv$, where u and v are nonadjacent vertices in G?

A5 The *bipartite Ramsey number* $BR(F, H)$ of two bipartite graphs F and H is the smallest positive integer r such that every red-blue coloring of $K_{r,r}$ results in a red F or a blue H. Show that $BR(P_5, P_5) = 5$.

Part B

B1 Prove, for every positive integer k, that the complete graph K_{6k+4} is 3-factorable, where each 3-factor is Hamiltonian.

B2 (a) Let B_1 be a bipartite graph with partite sets U_1 and W_1 where $|U_1| = |W_1| = n$. State Phillip Hall's Theorem in this setting that provides a necessary and sufficient condition for B_1 to have a perfect matching.

(b) Let B_2 be a different bipartite graph with partite sets U_2 and W_2 where $2|U_2| = |W_2| = 2n$. State another theorem similar to Phillip Hall's Theorem that will characterize all graphs that contain n vertex disjoint copies of P_3.

(c) Prove your new theorem.

B3 (a) Prove that if T is a strong tournament of order $n \geq 3$, then every arc of T lies on a cycle.

(b) Prove that if T is a strong tournament of order $n \geq 3$, then the arcs of T can be covered by at most $\binom{n-1}{2}$ cycles; that is, there is a collection of $\binom{n-1}{2}$ or fewer cycles such that every arc of T belong to at least one of these cycles.

(c) Prove that for each integer $n \geq 4$, there is a a strong tournament of order n whose arcs cannot be covered by fewer than $\binom{n-1}{2}$ cycles.

B4 Let m be the minimum size of a graph G of order $n = 8k + 2 \geq 10$ and independence number $\alpha(G) = 4k + 2$ for which G has a triangle. Show that $m = (4k + 1)^2$.

B5 (a) Let F and H be two nonempty graphs, where $x \in V(F)$ and $y \in V(H)$. Suppose that F' is isomorphic to $F - x$ and H' is isomorphic to $H - y$. Prove that

$$R(F, H) \leq R(F', H) + R(F, H').$$

State (without proof) conditions on $R(F', H)$ and $R(F, H')$ such that

$$R(F, H) \leq R(F', H) + R(F, H') - 1.$$

(b) Use (a) to give an upper bound for $R(K_3, K_4)$.

B6 Let G be a graph with no even cycles and order n.

(a) Prove that no odd cycle of G can contain a chord.

(b) For each $n \geq 1$, what is the maximum size of G?

(c) If G has odd order, no even cycles, and maximum size, what is the structure of G? Prove it.

(d) If G has even order, no even cycles, and maximum size, what is the structure of G? Prove it.