Graph Theory Preliminary Exam January 7, 2010

Instructions Do **exactly** four of the five problems in **Part A** and do **exactly** four of the five problems in **Part B**. Indicate clearly which problem in Part A and which problem in Part B you have omitted. Each problem in Part A is valued at 10 points, while each problem in Part B is valued at 15 points. Hand in **eight** problems only. Begin each problem on a new sheet of paper and write on one side of the paper only. You have six hours to complete the exam. When you are ready to hand in your exam, assemble your solutions in numerical order and write your name on the front page.

Part A

A1 (10 points) By a chorded cycle is meant a cycle C (of length at least 4) together with an edge that joins two non-consecutive vertices of C. Prove that every 3-connected graph contains a chorded cycle but that this need not be the case for a 2-connected graph.

A2 (10 points)

(a) Determine all self-complementary trees.

(b) Use the Four Color Theorem to show that every planar graph can be decomposed into two bipartite graphs.

A3 (10 points) For a positive integer k, let G be a $(2k+2)$-regular graph of order $4k+1$.

(a) Show that G is Hamiltonian

(b) Let $C : v_1, v_2, \ldots, v_{4k+1}, v_{4k+2} = v_1$ be a Hamiltonian cycle of G and let $X = \{v_{2i-1}v_{2i} : 1 \leq i \leq 2k+1\}$. Prove that $H = G - X$ is of Class two, that is, the edge chromatic number is $\chi_1(H) = \Delta(H) + 1$.

A4 (10 points) Let T be a tournament with strong components $S_1, S_2, \ldots, S_p$ where $V(S_i) = V_i$ for $1 \leq i \leq p$ such that if $u \in V_i$ and $v \in V_j$, where $1 \leq i < j \leq p$, then (u, v) is an arc of T.

(a) It is known that every tournament contains a Hamiltonian path. Let P be a Hamiltonian path of T. Show that the subdigraph of P induced by V_i is a Hamiltonian path of S_i for each i with $1 \leq i \leq p$.

(b) Prove that if T is a tournament of order $\binom{k}{2}$ for some integer $k \geq 3$ in which no two strong components have the same order, then T contains at least k Hamiltonian paths.

A5 (10 points) Let G be a bipartite graph of size m with partite sets V_1 and V_2 where $|V_1| = |V_2| = k$.

(a) Prove that if $m \geq k^2 - k + 1$, then G must have a perfect matching.

(b) Show that for $m = k^2 - k$, there exists a bipartite graph H with partite sets V_1 and V_2 such that $|V_1| = |V_2| = k$ but H does not have a perfect matching.

Part B

B1 (15 points) If G is a graph of order n such that $\chi(G) = n$, then $\chi(G) = \omega(G)$.

(a) Show that if G is a graph of order $n \geq 2$ such that $\chi(G) = n - 1$, then $\chi(G) = \omega(G)$.

(b) Show that if G is a graph of order n for some $n \geq 3$ such that $\chi(G) = n - 2$, then it need not be the case that $\chi(G) = \omega(G)$.

B2 (15 points) A graph G is obtained from $K_{4,4}$ by removing four independent edges. Let $V(G) = \{v_1, v_2, \ldots, v_8\}$. The 8-tuple $(\pi_1, \pi_2, \ldots, \pi_8)$ of cyclic permutations defined by

$$\pi_1 = (286) \quad \pi_2 = (153) \quad \pi_3 = (248) \quad \pi_4 = (357)$$
$$\pi_5 = (264) \quad \pi_6 = (751) \quad \pi_7 = (468) \quad \pi_8 = (173)$$

describes a 2-cell embedding of G on some surface S_k.

(a) What is k? Explain.

(b) Is $k = \gamma(G)$? Explain.

(c) Is $k = \gamma_M(G)$? Explain.

B3 (15 points)

(a) Prove that if a connected cubic graph has a 1-factor F and a bridge e, then F must include the bridge e.

(b) Suppose that a connected cubic graph G has a minimum edge-cut consisting of two edges e and f. What can we conclude about any 1-factor of G? May it contain neither e nor f? May it contain both e and f? May it contain exactly one of e and f?

B4 (15 points) A unicyclic graph is a connected graph with exactly one cycle. Prove that a sequence $s_n : d_1, d_2, \ldots, d_n$ $(n \geq 3)$ of integers with $1 \leq d_i \leq n - 1$ for $1 \leq i \leq n$ is a degree sequence of a unicyclic graph of order n if and only if at most $n - 3$ terms of s_n are 1 and $\sum_{i=1}^{n} d_i = 2n$.

B5 (15 points) Let $r = r(K_{n_1}, K_{n_2})$ be the classic Ramsey number of two complete graphs.

(a) Let T_s be any tree of order $s \geq 2$. Show that the three color generalized Ramsey number $r(K_{n_1}, K_{n_2}, T_s)$ is given by the formula $r(K_{n_1}, K_{n_2}, T_s) = 1 + (r-1)(s-1)$.

(b) How does this result generalize to $k \geq 3$ complete graphs plus one tree?