Graph Theory Preliminary Examination
January 7, 2014

Instructions

Do **exactly four** of the five problems in **Part A** and do **exactly four** of the six problems in **Part B**. Indicate clearly which problem in Part A and which problems in Part B you have omitted. Each problem in Part A is valued at 10 points, while each problem in Part B is valued at 15 points.

Hand in **eight** problems only. Begin your solution of each problem on a new sheet of paper and write on one side of the paper only. You have six hours to complete the exam.

When you are ready to hand in your exam, assemble your solutions in numerical order and write your name on the front page.

Part A

A1 Prove that there exists no k-chromatic graph G of order k^2+1 $(k \geq 3)$ for which $\chi(\overline{G}) = \chi(G)$.

A2 Let G be a connected graph such that the length of a longest path in G is ℓ.

(a) Prove that no two paths of length ℓ in G are vertex-disjoint.

(b) By (a), two paths of length ℓ cannot be vertex-disjoint. Prove that if P and Q are two paths of length ℓ that meet in a single vertex, then ℓ is even.

A3 Let G be a k-connected graph, $k \geq 3$, and let $v, v_1, v_2, \ldots, v_{k-1}$ be k vertices of G. Show that G has a cycle containing all of $v_1, v_2, \ldots, v_{k-1}$ but not v and $k-1$ internally disjoint $v - u_i$ paths P_i $(1 \leq i \leq k-1)$ such that for each i, the vertex u_i is the only vertex of P_i on C.

A4 Prove for every planar graph G of order 3 or more that there exists a partition $\{V_1, V_2, V_3\}$ of $V(G)$ such that each subgraph $G_i = G[V_i]$ $(i = 1, 2, 3\ )$ induced by V_i in G is a forest.

A5 (a) Prove that for every positive integer n, there exists positive integer N such that if X is an N-element set and the set $\binom{X}{2}$ (all pairs) is partitioned into two sets E_1 and E_2, then at least one of the graphs (X, E_1) and (X, E_2) contains a complete subgraph on n vertices.

(b) Generalize (a) for a partition of $\binom{X}{2}$ into r parts, and prove this generalization.

Part B

B1 Prove that an Eulerian graph G can only be decomposed into odd cycles if and only if each block of G is an odd cycle.

B2 Let G be a graph of order at least 3. Prove or disprove:

(a) If G is maximal planar, then G must be Hamiltonian.

(b) If G is maximal toroidal, then G must be Hamiltonian.

A graph G is *toroidal* if G can be embedded in the torus.

B3 The *bipartite Ramsey number* $BR(H, H)$ of a bipartite graph H is defined as the smallest number n such that any 2-coloring of the complete bipartite graph $K_{n,n}$ must contain a monochromatic copy of H. Prove that $BR(C_4, C_4) = 5$.

B4 In any graph G with maximum independent edge set of size $\alpha'(G)$, for each i with $0 \leq i \leq \alpha'(G)$, let a_i denote the number of independent edge sets of size i. Recall that the corona $cor(H)$ of a graph H is formed by adding a pendant edge at each vertex of H. In the graph $cor(K_n)$, prove that the number of independent edge sets of each size satisfy $a_i = a_{n-i}$.

B5 There are $n \geq 2$ men armed with squirt guns and standing about in the plane in such a manner that none of the $\binom{n}{2}$ distances is duplicated. All distances are distinct. At a signal, each man squirts his (unique) closest neighbor.

(a) For which values of n can it happen that everyone gets wet?

(b) For which values of n is it guaranteed that someone remains dry?

(c) Prove your result for every value of $n \geq 2$.

B6 Let G be a weighted graph in which each vertex initially has weight 1. A *total acquisition move* transfers all of the weight from a vertex u to a neighboring vertex v, under the condition that before the move the weight on v is at least as large as the weight on u. The *total acquisition number* of G, written $a_t(G)$, is the minimum size of the set of vertices with positive weight after a sequence of total acquisition moves.

(a) Show that $a_t(G) \leq \alpha(G)$ and $a_t(G) \leq \gamma(G)$, where $\alpha(G)$ denotes the maximum size of an independent set in G and $\gamma(G)$ denote the minimum size of a dominating set in G.

(b) Prove that for any two graphs G_1 and G_2 with disjoint vertex sets,

$$a_t(G_1 \square G_2) \leq a_t(G_1) \cdot a_t(G_2),$$

where $\square$ stands for the Cartesian product.

(c) Show that $a_t(Q_n) = 1$ for the n-cube Q_n where $n \geq 2$.