Graph Theory Preliminary Exam 2009

Instructions Do **exactly** four of the five problems in **Part A** and do **exactly** four of the five problems in **Part B**. Indicate clearly which problem in Part A and which problem in Part B you have omitted. Each problem in Part A is valued at 10 points, while each problem in Part B is valued at 15 points. Hand in **eight** problems only. Begin each problem on a new sheet of paper and write on one side of the paper only. You have six hours to complete the exam. When you are ready to hand in your exam, assemble your solutions in numerical order, write your name on the front page, and initial all other pages.

Part A

A1 (10 points)

(a) Prove that every Hamiltonian-connected graph of order 4 or more is 3-connected.

(b) Let G be a graph of order $n \geq 3$ having the property that for each $v \in V(G)$, there is a Hamiltonian path with initial vertex v. Show that G is 2-connected but not necessarily Hamiltonian.

(c) Let G be a graph of order $n \geq 3$ such that $\deg v \geq n/2$ for every vertex v of G. Show that if v_1, v_2, and v_3 are any three vertices of G, there exists a Hamiltonian cycle C of G containing v_1, v_2, v_3 in the order given.

A2 (10 points) Determine all graphs G of order $n \geq 5$ and size $m = 3n - 5$ such that for each edge e of G, the graph $G - e$ is planar.

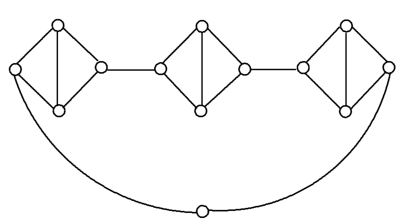
A3 (10 points) Recall that the Ramsey number $r(K_3, K_4) = 9$. Prove that $r(K_3, C_5) = 9$ as well.

A4 (10 points) For a tournament T of order n, let

$$\Delta = \max\{\operatorname{od} v : v \in V(T)\} \text{ and } \delta = \min\{\operatorname{od} v : v \in V(T)\}.$$

Prove that if $\Delta - \delta < \frac{n}{2}$, then T must be strong.

A5 (10 points) Recall that the total chromatic number $\chi_2(G)$ is the fewest number of colors needed to color all the vertices and edges of a graph so that adjacent vertices, adjacent edges, and incident vertices and edges must receive different colors. Find $\chi_2(G)$ for this graph.



Part B

B1 (15 points) For a strong digraph D, let $\vec{d}(u,v)$ be the directed distance from u to v (the length of a shortest directed $u-v$ path in D). Define the directed eccentricity $e(v)$ of v by

$$e(v) = \max\left\{\frac{\vec{d}(x,v)+\vec{d}(v,y)}{2} : x, y \in V(D)\right\}$$

and define the radius and diameter of D by

$$\text{rad}(D) = \min\{e(v) : v \in V(D)\} \text{ and } \text{diam}(D) = \max\{e(v) : v \in V(D)\}.$$

Define the center $Cen(D)$ of D as the subdigraph of D induced by those vertices of D with eccentricity $\text{rad}(D)$.

(a) Prove, for every strong digraph D, that

$$\text{rad}(D) \leq \text{diam}(D) \leq 2\,\text{rad}(D).$$

(b) Prove, for every oriented graph D', that there exists a strong oriented graph D such that $Cen(D) = D'$.

B2 (15 points)

(a) Prove that every Eulerian graph of odd order has three vertices of the same degree.

(b) Prove that if G is an Eulerian graph of odd order, then so is $\overline{G} + K_2$.

(c) Prove that for each odd integer $n \geq 3$, there exists exactly one Eulerian graph of order n containing exactly three vertices of the same degree and at most two vertices of any other degree.

B3 (15 points) For integers s and t with $1 \leq s \leq t$, define the Ramsey chromatic number $r_\chi(s,t)$ to be the smallest positive integer n such that for any 2-coloring of the edges of K_n with the colors blue and red, either the blue subgraph has chromatic number at least s, or the red subgraph has chromatic number at least t. Trivially, $r_\chi(1,t) = 1$ and $r_\chi(2,t) = t$.

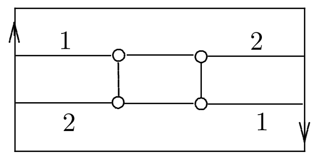
(a) Does $r_\chi(s,t)$ exist for all integers $1 \leq s \leq t$?

(b) Prove that $r_\chi(s,t) \leq r_\chi(s-1,t) + r_\chi(s,t-1)$.

(c) Determine $r_\chi(3,3)$.

(d) For $3 \leq s \leq t$, determine with proof a simple formula for $r_\chi(s,t) - 1$ in terms of s and t.

B4 (15 points) A rectangle can be used to represent graphs embedded on a Möbius band by identifying the opposite vertical sides with a twist. For example, the drawing shown here is viewed as a 2-cell embedding of K_4 on the Möbius band. The single region bordering the single boundary of the band, both top and bottom, is allowed as a region even though it is technically not a 2-cell. You may use the fact that on the Möbius band, Euler's polyhedral formula becomes $n - m + r = 1$.



(a) For each $n \geq 3$, find the maximum number of edges in a graph drawn on the Möbius band.

(b) If a graph G drawn on the Möbius band has order $n \geq 3$ and girth g, what is the maximum number of edges in G?

(c) Can the Petersen graph be drawn on the Möbius band?

(d) Which complete bipartite graphs $K_{s,t}$ with $1 \leq s \leq t$ can be drawn on the Möbius band?

B5 (15 points) Let G be a graph with no even cycles and order n.

(a) Prove that no odd cycle of G can contain a chord.

(b) For each $n \geq 1$, what is the maximum size of G?

(c) If G has odd order, no even cycles, and maximum size, what is the structure of G? Prove it.

(d) If G has even order, no even cycles, and maximum size, what is the structure of G? Prove it.