Graph Theory Preliminary Exam 2011

Instructions Do **exactly** four of the five problems in **Part A** and do **exactly** four of the five problems in **Part B**. Indicate clearly which problem in Part A and which problem in Part B you have omitted. Each problem in Part A is valued at 10 points, while each problem in Part B is valued at 15 points. Hand in **eight** problems only. Begin each problem on a new sheet of paper and write on one side of the paper only. You have six hours to complete the exam. When you are ready to hand in your exam, assemble your solutions in numerical order and write your name on the front page.

Part A

A1 Prove that if the diameter of a connected graph G is at least 3, then the diameter of its complement $\overline{G}$ is at most 3.

A2 For each positive integer k, let G_k and H_k be the two graphs of order 2k + 3 where $G_k = kK_2 + K_3$ (the join of kK_2 and K_3) and $H_k = \overline{kK_2} + K_3$ (the join of $\overline{kK_2}$ and K_3). Determine:

(a) all values of k for which G_k is Eulerian.

(b) all values of k for which G_k is Hamiltonian.

(c) the chromatic index (the edge chromatic number) of H_k for each k.

A3 Determine the genus of the Cartesian product $K_4 \times K_2$.

A4 Determine the Ramsey number $r(F, H)$ where $F = H$ is shown in Figure 1.

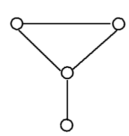

Figure 1: The graph $F = H$

A5 Let G be a bipartite graph of size m with partite sets V_1 and V_2 where $|V_1| = |V_2| = k$.

(a) Prove that if $m \geq k^2 - k + 1$, then G must have a perfect matching.

(b) Prove or disprove: the lower bound in (a) is sharp.

Part B

B1 A graph G is *vertex transitive* if for every pair u, v of vertices of G, there exists an automorphism α such that $\alpha(u) = v$.

(a) If a self-complementary graph G is also vertex transitive, then what restriction does this place on the possible order of G?

(b) Show that there exists self-complementary vertex transitive graphs of arbitrarily large order. You need to show how to find such graphs of arbitrarily large order, but you do not have to find one such graph for each order permitted in (a).

B2 Prove or disprove: There exists no 3-chromatic graph G such that $\chi(G-v) = 3$ for every vertex v of G but $\chi(G-x-y) = 2$ for every two vertices x and y of G.

B3 Show that every graph of order 10 and size 34 either contains K_5 as a subgraph or contains $K_3 \cup K_4$ as a subgraph.

B4 Let G be a graph of order n. Determine, with a proof, each of the following:

(a) the maximum number of edges in G such that G has no odd cycle.

(b) the maximum number of edges in G such that G has no even cycle.

B5 Prove or disprove: If G is a connected graph of even size m, then G can be decomposed into $\frac{m}{2}$ copies of P_3.