Graph Theory Preliminary Exam 2013

Instructions

Do **exactly four** of the five problems in **Part A** and do **exactly four** of the six problems in **Part B**. Indicate clearly which problem in Part A and which problems in Part B you have omitted. Each problem in Part A is valued at 10 points, while each problem in Part B is valued at 15 points.

Hand in **eight** problems only. Begin your solution of each problem on a new sheet of paper and write on one side of the paper only. You have six hours to complete the exam.

When you are ready to hand in your exam, assemble your solutions in numerical order and write your name on the front page.

Part A

A1 Let G be a connected graph of order $n \geq 3$ where each vertex of G is colored with one of $1, 2, 3$ (not necessarily a proper coloring) and each color is used at least once. Prove that G contains a path on which there are three vertices with distinct colors.

A2 Let G be a k-connected graph, $k \geq 3$, and let $v, v_1, v_2, \ldots, v_{k-1}$ be k vertices of G. Show that G has a cycle containing all of $v_1, v_2, \ldots, v_{k-1}$ but not v and $k-1$ internally disjoint $v - u_i$ paths P_i $(1 \leq i \leq k-1)$ such that for each i, the vertex u_i is the only vertex of P_i on C.

A3 For $i = 1, 2, \ldots, 5$, let $G_i = P_4$. A graph G of order n is obtained from the graphs $G_1, G_2, \ldots, G_5, G_6 = G_1$ by joining each vertex of G_i to each vertex of G_{i+1} for $i = 1, 2, \ldots, 5$.

(a) Determine the vertex independence number $\alpha(G)$ and the clique number $\omega(G)$ of the graph G. Each of the numbers $n/\alpha(G)$, $n + 1 - \alpha(G)$ and $\omega(G)$ is a bound for the chromatic number $\chi(G)$. Evaluate each bound and indicate what each of these bounds says about $\chi(G)$.

(b) Determine $\chi(G)$ with explanation.

(c) Use (a) and (b) to determine whether G is a perfect graph.

(d) Is the complement $\overline{G}$ of G a perfect graph? Explain.

A4 Shown in Figure 1 is a 6-regular graph G of order 14. Answer each of the following questions with explanation.

(a) Is G 1-factorable?

(b) Is G 2-factorable?

(c) Is G Hamiltonian-factorable?

(d) Is G K_3-decomposable?

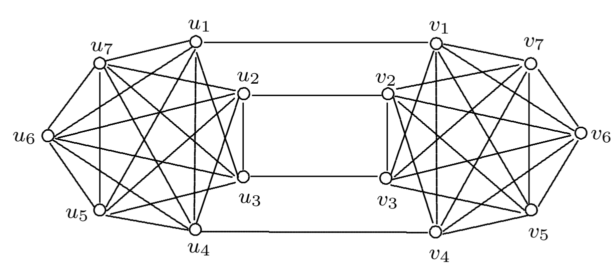

Figure 1: A 6-regular graph of order 14

A5 Prove that if T is a tournament of order $4r$ with $r \geq 1$, where $2r$ vertices of T have outdegree $2r$ and the other $2r$ vertices have outdegree $2r - 1$, then T is strong.

Part B

B1 Let $s : d_1, d_2, \cdots, d_n$ be a non-increasing sequence of $n \geq 2$ positive integers, that is, $d_1 \geq d_2 \geq \cdots \geq d_n \geq 1$. Characterize those sequences s that are degree sequences of some tree of order n. That is, state and prove a theorem of the form:

> The sequence s is the degree sequence of some tree of order n if and only if ....

B2 Determine all graphs G of order $n \geq 5$ and size $m = 3n - 5$ such that for each edge e of G, the graph $G - e$ is planar.

B3 For positive integers a and b, determine (with proof) the Ramsey number $r(K_{1,2a}, K_{1,2b})$.

B4
- (a) Let B_1 be a bipartite graph with partite sets U_1 and W_1 where $|U_1| = |W_1| = n$. State Phillip Hall's Theorem in this setting that tell us a necessary and sufficient condition for B_1 to have a perfect matching.
- (b) Let B_2 be a different bipartite graph with partite sets U_2 and W_2 where $2|U_2| = |W_2| = 2n$. State another theorem similar to Phillip Hall's Theorem that will characterize all graphs that contain n vertex disjoint copies of P_3.
- (c) Prove your new theorem.

B5 A tree T is a caterpillar if T does not contain the subdivision $S(K_{1,3})$ as shown in Figure 2. For any nontrivial tree T of order at least 3, let T' be the tree obtained by removing all end-vertices from T. Prove that the following:

- (a) If T is a caterpillar, then T' is a path.
- (b) The square T^2 of a tree T is Hamiltonian if and only if T is a caterpillar.

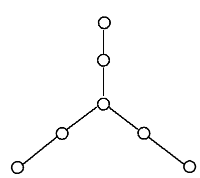

Figure 2: The graph $S(K_{1,3})$

B6 For an integer $k \geq 2$, a graph G is *uniquely k-colorable* if $\chi(G) = k$ and every two proper k-colorings of G have the same color classes. That is, G has a unique k-coloring (except for the names of colors).

- (a) Suppose that c is a unique k-coloring of G resulting in color classes $V_1, V_2, \ldots, V_k$. Show that the subgraph $G[V_i \cup V_j]$ induced by V_i and V_j is connected for all i, j with $1 \leq i < j \leq k$.
- (b) Show that if G is a maximal planar graph of order $n \geq 3$ such that $\chi(G) = 3$, then G is uniquely 3-colorable. Is the converse of (b) true? Explain.
- (c) Let G be a planar and uniquely 4-colorable graph. Use (a) to show that G is maximal planar.