Graph Theory Preliminary Examination

May 30, 2015

Instructions

Do **exactly four** of the six problems in **Part A** and do **exactly four** of the six problems in **Part B**. Indicate clearly which two problems in Part A and which two problems in Part B you have omitted. Each problem in Part A is valued at 10 points, while each problem in Part B is valued at 15 points.

Hand in **eight** problems only. Begin your solution of each problem on a new sheet of paper and write on one side of the paper only. You have six hours to complete the exam.

When you are ready to hand in your exam, assemble your solutions in numerical order and write your name on the front page.

Part A

A1 Let G be a graph of order n and size m. Prove that G is a tree if and only of $m = n-1$ and $q \leq p-1$ for every induced subgraph of order p and size q.

A2 Let G be a connected cubic plane graph of order n and size m having r regions such that the boundary of each region is either a 5-cycle or a 6-cycle. Determine the number of regions whose boundary is a 5-cycle.

A3 Let G be a graph of order $n = 2k \geq 6$ such that each vertex of G has degree $k-1$ or $k+1$, where at least k vertices have degree $k+1$. Prove that G is Hamiltonian.

A4 Let G be a graph of odd order $2k+1 \geq 5$ such that $2k$ of the vertices of G have the same degree r and the remaining vertex of G has degree s for positive integers r, s with $r > s$. Determine whether G is of Class one or Class two.

A5 Prove: If G is a k-connected graph, $k \geq 2$, then every k vertices of G lie on a common cycle of G.

[You can use the fact (without providing a proof) that if G is a k-connected graph, $k \geq 2$, and $u, v_1, v_2, \ldots, v_t$ are $t+1$ distinct vertices of G, where $2 \leq t \leq k$, then G contains a $u - v_i$ path for each i ($1 \leq i \leq t$), every two paths of which have only u in common.]

A6 (a) Prove that if T is a strong tournament of order $n \geq 3$, then every arc of T lies on a cycle.

(b) Prove that if T is a strong tournament of order $n \geq 3$, then the arcs of T can be covered by at most $\binom{n-1}{2}$ cycles; that is, there is a collection of $\binom{n-1}{2}$ or fewer cycles such that every arc of T belong to at least one of these cycles.

(c) Prove that for each integer $n \geq 4$, there is a a strong tournament of order n whose arcs cannot be covered by fewer than $\binom{n-1}{2}$ cycles.

Part B

B1 Prove, for every integer $n \geq 22$ such that $n \equiv 10 \pmod{12}$, that the complete graph K_n is 3-factorable, where each 3-factor is Hamiltonian.

B2 A graph G of order 14 and size 48 is 2-cell embedded on the double torus (the surface of genus 2).

(a) How many regions are there in this embedding? Explain.

(b) Can G be embedded on the torus?

(c) What is the genus $\gamma(G)$ of G?

(d) Let u and v be two nonadjacent vertices in G. What is the genus $\gamma(G + uv)$ of $G + uv$?

B3 Recall that the Nordhaus-Gaddum Theorem says, in part, that if G is a graph of order n, then $\chi(G)\chi(\overline{G}) \geq n$.

(a) Prove that if the complete graph K_n is decomposed into three nonempty graphs G_1, G_2 and G_3, then $\chi(G_1)\chi(G_2)\chi(G_3) \geq n$.

(b) Prove that there are infinitely many integers n for which K_n can be decomposed into three nonempty graphs G_1, G_2 and G_3 such that $\chi(G_1)\chi(G_2)\chi(G_3) = n$. [Hint: $n = t^3$ for some integer $t \geq 2$ may be a convenient choice.]

B4 For graphs F and H, let $R_2(F, H) = n$ means that every red-blue coloring of the complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ of order n results in either a red H or a blue F but there is a red-blue coloring of the complete bipartite graph $K_{\lfloor (n-1)/2 \rfloor, \lceil (n-1)/2 \rceil}$ of order $n - 1$ that avoids both a red H and a blue F.

(a) Does $R_2(F, H)$ exist for every pair of graph F, H or do there exist pairs of F, H for which $R_2(F, H)$ does not exist?

(b) Prove that $R_2(C_4, C_4) = 10$.

B5 Determine the minimum size m of a graph G of order $n = 8k+2 \geq 10$ and independence number $\alpha(G) = 4k + 1 + a$ where $1 \leq a \leq 4k$ for which G must have a triangle.

B6 For each integer $k \geq 2$, prove that the minimum positive integer n such that every k-edge (non-proper) coloring of K_n results in a monochromatic subgraph G of K_n with $\chi(G) \geq 3$ is $n = 2^k + 1$.