WMU Department of Mathematics
Algebra Comprehensive Exam
August 28, 2015

Instructions. Work all of these problems and write their solutions clearly and completely (and legibly). Write your solution to each problem on a separate sheet of paper, with your name at the top of each page. You have 6 hours to complete this exam.

1. Let $G = AB$ be the semi-direct product of a group A by a group B (i. e., $B \lhd G$, $G = AB$, and $A \cap B = \{e\}$). Let $N = [A, B]$ denote the subgroup of G generated by all commutators $[a, b] = aba^{-1}b^{-1}$, where $a \in A$ and $b \in B$. (a) Prove that N is a normal subgroup of G contained in B. (b) Prove that $G/N \cong A \times (B/N)$.

2. Let G be a non-abelian group of order pq, with p, q prime and $p < q$. (a) Prove that p divides $q - 1$. (b) Prove that the center of G is trivial. (c) How many distinct conjugacy classes are there in G?

3. Find the elementary divisors, invariant factors, minimal polynomial, characteristic polynomial, rational canonical form, and the Jordan canonical form of the $\mathbb{Q}$-matrix

$$A = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 4 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

4. Let $F \subset E \subset L$ be fields. Show that if L is algebraic over E and E is algebraic over F then L is algebraic over F.

5. Let R be a ring with 1. Suppose an element $r \in R$ has a one-sided inverse. (a) If R is finite, show that r has a two-sided inverse. (Hint: Consider right or left multiplication by r.) (b) Show that the conclusion is false without the finiteness hypothesis.

6. Let $R = \mathbb{Q}[x]/(x^2+1)^2$. Classify all finitely-generated R-modules. (Hint: Every R-module is a $\mathbb{Q}[x]$-module.)

7. Exhibit 5 rings of order 4, no two of which are isomorphic. Prove that no two of them are isomorphic.