**WMU Department of Mathematics**
**Algebra Comprehensive Exam**
**March 11, 2016**

**Instructions.** This exam has two sections, "Required Problems" and "Elective Problems". You must work all of the required problems and 3 of the elective problems. **It is your responsibility to declare which of your solutions to elective problems should be considered by your examiners.**

Write your solution to each problem on a separate sheet of paper, with your name at the top of each page. Please write clearly and legibly. You have 6 hours to complete this exam.

## Required Problems.

1. Let $f : R \to S$ be a homomorphism of commutative rings and $I \subseteq S$ be an ideal. Show that
(a) $f^{-1}(I)$ is an ideal of $R$,
(b) the induced map $R/f^{-1}(I) \to S/I$ is injective, and
(c) if $I$ is prime, then so is $f^{-1}(I)$.

2. Let $V$ be a finite dimensional vector space over $\mathbb{Q}$ and suppose $T$ is a nonsingular linear transformation of $V$ such that $T^{-1} = T^2 + T$.
(a) Prove that the dimension of $V$ is divisible by 3.
(b) Assume that the dimension of $V$ is precisely 3. Prove that all such transformations $T$ are similar.

3. Let $F = \mathbb{Q}\left(\sqrt{2}, i\right)$ and $K = \mathbb{Q}\left(\sqrt{2}, \sqrt[3]{2}\right)$.
(a) Prove that $F/\mathbb{Q}$ is a normal extension over $\mathbb{Q}$.
(b) Determine the Galois group of $F/\mathbb{Q}$.
(c) Exhibit all intermediate extensions between $F$ and $\mathbb{Q}$.
(d) Show that $K/\mathbb{Q}$ is not a normal extension.
(e) Compute $[K : \mathbb{Q}]$.

4. Let $R$ be a ring.
(a) Assuming $M$ is a simple $R$-module, show that $\mathrm{End}_R(M)$ is a division ring.
(b) Assuming $M$ and $N$ are simple $R$-modules with $M \ncong N$, show that $\mathrm{Hom}_R(M, N) = \{0\}$.

## Elective Problems.

5. Suppose $T$ and $U$ are subgroups of a group $G$, $G = T \ltimes U$ is a semidirect product, and $H$ is a subgroup of $G$ containing $U$. Prove that $H = (H \cap T) \ltimes U$.

6. Suppose that a $p$-group $G$ acts on a finite set $S$. Show that the number of fixed points of the action is congruent mod $p$ to the cardinality $|S|$ of the set $S$.

7. Let $G$ be a finite simple group of order divisible by $p^2$, where $p$ is prime. Show that every proper subgroup of $G$ has index greater than $p$.

8. Prove that the alternating group $A_n$ contains precisely two conjugacy classes of $n$-cycles whenever $n \in \mathbb{N}$ is odd and greater than 1.

9. Suppose $0 \to A \to B \to \mathbb{Z}^n \to 0$ is an exact sequence of abelian groups. Show that $B \cong A \times \mathbb{Z}^n$.