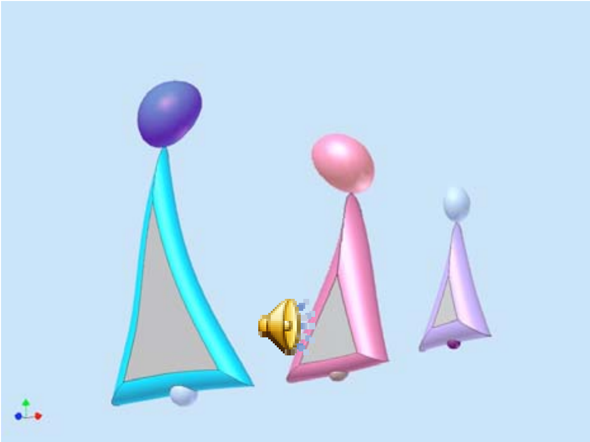
Understanding Proportion–II Module

Thinking and Modeling Multiplicatively



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Technical Issues

- Before you start the slides, please install LiveWeb to activate the embedded applets.

- For PowerPoint 97–2003 users:
  Go to the link http://skp.mvps.org/liveweb.htm and download http://skp.mvps.org/downloads/liveweb.zip (zip file).

  For PowerPoint 2007 users: Go to the same link and download http://skp.mvps.org/downloads/livewebfor2007.zip (zip file).

- **Notice**: If you use a Mac, unfortunately the embedded applets will not be activated, please go to the links given in the related pages.

- Adobe Acrobat is also required for many of the modules: Go to the link and download http://get.adobe.com/reader/

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Understanding Proportion–II

- **Introduction**: Goals, Objectives, Content Expectations
- **Lesson 1**: Indirect Measurement
- **Lesson 2**: Candle Burning Tasks
- **Reflections and Evaluation**

Introduction:
Purpose of Module

- Engage teachers by utilizing proportional reasoning in various contexts
- Provide teachers with pedagogical knowledge for proportional reasoning concepts leading to greater understanding of the concepts and improved skills by their students

Objectives of Understanding Proportion – II Session

- Explore two different contexts for proportional reasoning: Indirect Measurement, and Candle Burning
- Apply geometric, numeric, statistical, and algebraic reasoning when exploring proportional contexts

Relevant Content Expectations

- Michigan Content Expectations relevant to this module on understanding proportion are presented in the next two slides.
- Each slide corresponds to one of the proportional reasoning contexts explored in this session.
- Take a few minutes to review these content expectations.

Michigan HSCEs Related to Indirect Measurement

- G.2.3.4 Use theorems about similar triangles to solve problems with and without use of coordinates.

Michigan GLCEs Related to Candle Burning

- A.PA.07.03 Given a directly proportional or other linear situation, graph and interpret the slope and intercept(s) in terms of the original situation; evaluate $y = mx + b$ for specific x values.
- A.PA.07.04 For directly proportional or linear situations, solve applied problems using graphs and equations.
- A.PA.07.05 Recognize and use directly proportional relationships of the form $y = mx$, and distinguish from linear relationships of the form $y = mx + b$, b non-zero; understand that in a directly proportional relationship between two quantities, one quantity is a constant multiple of the other quantity.
- A.PA.07.09 Recognize inversely proportional relationships in contextual situations; know that quantities are inversely proportional if their product is constant, and that the inversely proportional relationship is of the form k/x where k is some non-zero number.
- A.RP.07.10 Know that the graph of $y=k/x$ is not linear, know its shape, and know that it crosses neither the x-axis nor the y-axis.

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Lesson 1: Indirect Measurement

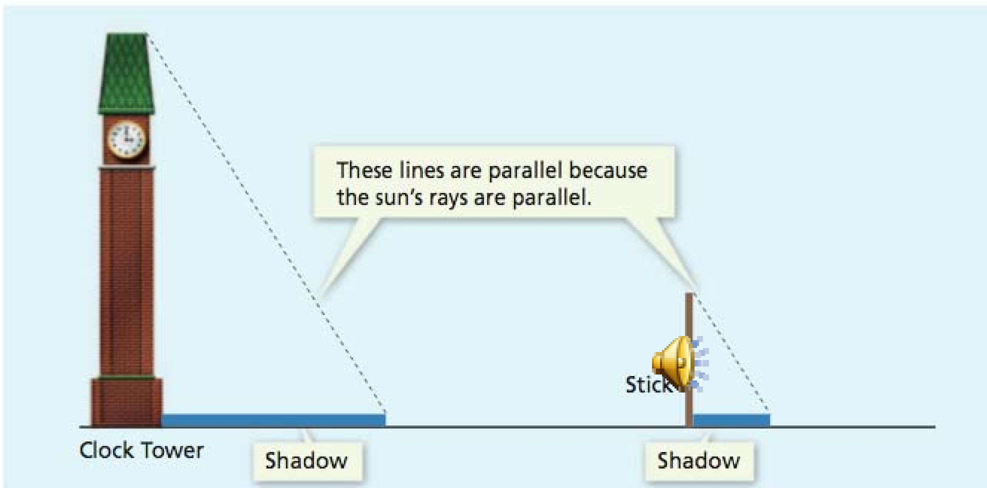
Indirect measurement is a technique that uses proportional reasoning to find a measurement when direct measurement is not possible.

In this section, you will investigate the following:

1. How can similarity in figures be used to measure hard-to-reach lengths or distances?

2. What makes indirect measurement difficult for students to understand?

Examine this image taken from page 78 of the Connected Mathematics Project 2 textbook, *Stretching and Shrinking.* There are two similar triangles depicted in the image. *How do we know that the two triangles are similar?*



A triangle is formed by a clock tower, its shadow, and an imaginary line from the top of the tower to the end of the shadow.

A triangle is formed by a stick, its shadow, and an imaginary line from the top of the stick to the end of its shadow.

78 Stretching and Shrinking

Click here for an answer

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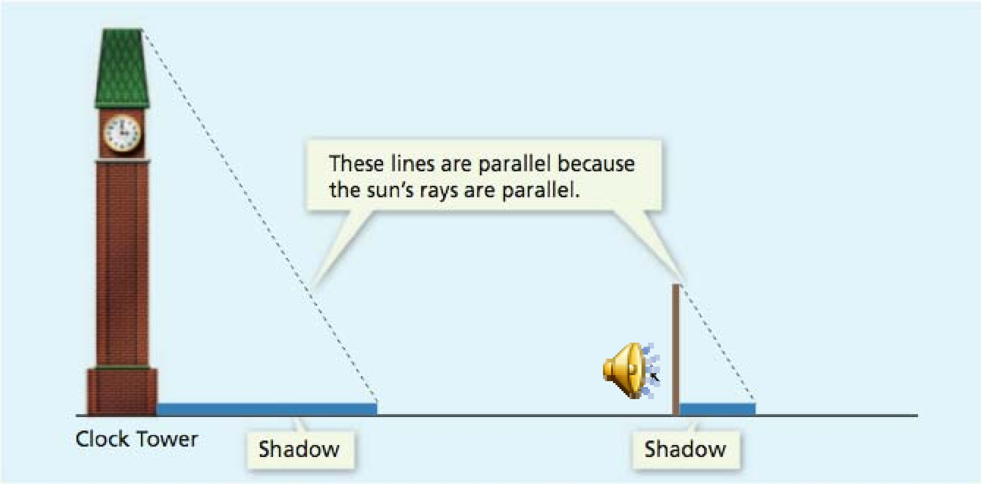
The triangles are similar because the corresponding angles all have the same measure. We know this is true because both the clock and the stick are perpendicular to the ground (right triangles) and because the dotted lines are parallel.



A triangle is formed by a clock tower, its shadow, and an imaginary line from the top of the tower to the end of the shadow.

A triangle is formed by a stick, its shadow, and an imaginary line from the top of the stick to the end of its shadow.

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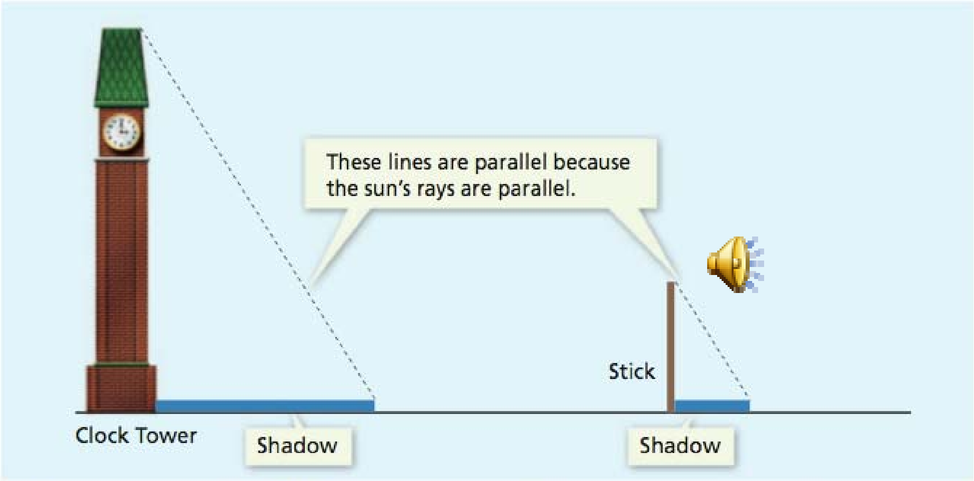
Shadow Task: Using similarity and proportion, measure the clock tower indirectly by using the following information:

- length of the stick: 3 meters
- length of the stick's shadow: 1.5 meters
- length of the building's shadow: 8 meters



A triangle is formed by a clock tower, its shadow, and an imaginary line from the top of the tower to the end of the shadow.

A triangle is formed by a stick, its shadow, and an imaginary line from the top of the stick to the end of its shadow.

How tall is the clock tower?

Find the solution to this problem and compare it to others' solutions. Then share your solution process with the group.

The clock tower in this problem is 16 meters tall. There are different ways to reason about this problem. Compare and contrast the following perspectives taken by Verna, Chris, and Amy and discuss them with your colleagues.

Verna	Chris	Amy
Since the stick's shadow is half of the length of the stick, the building's shadow should be half of the length of the building. Since the shadow is 8 meters, the building is 16 meters tall.	5 and 1/3 copies of the stick's shadow fit inside the clock tower's shadow. That means that 5 1/3 copies of the stick should fit inside the clock tower. 3 meters times 5 1/3 is 16 meters.	The stick is 1.5 meters taller than it's shadow is long. That means the clock tower is 1.5 meters longer than it's shadow.

You probably talked about many of the differences between the three students' perspectives on the shadow problem. Although they were not all correct in solving the problem, they do have something in common—they were all able to find the triangles and interpret the correspondences within the diagram.

Comments on Complexity

- The triangles were drawn on the task itself
- It was clear from the context which sides corresponded to one another.
- There were no other parts of the diagram to clutter the representation.

Recognizing the relevant correspondences isn't always a simple task—and may interfere with a student's ability to apply proportional reasoning to tasks like these. Keep these ideas in mind as you solve the next two tasks (hour glass and water slide).

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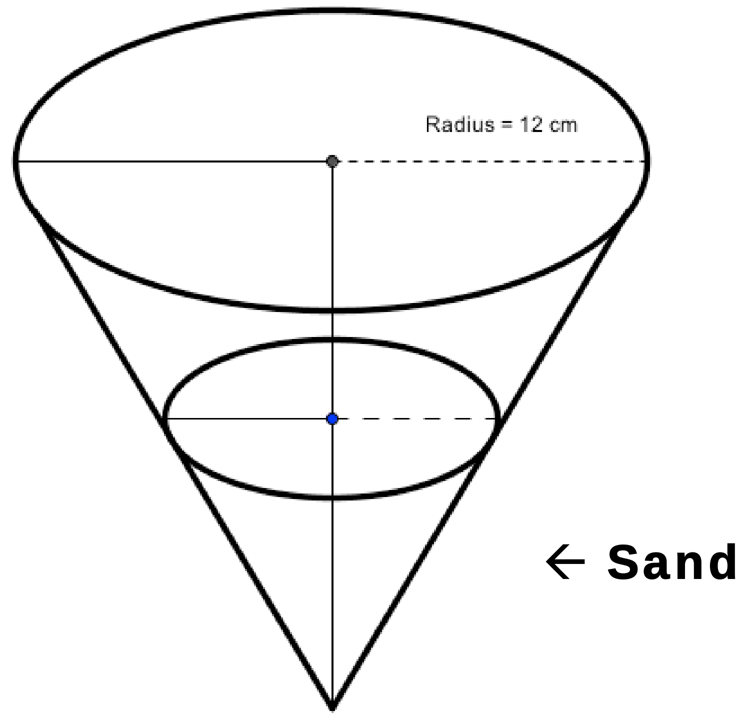
Hour Glass Task

A cone-shaped hourglass drips sand from the bottom point. When it is filled up with sand, the radius of the base is 12 cm and the height is 15 cm. After the sand drops for 12 seconds, the height of the sand decreases by 6 cm. What is the radius of the base at this time?

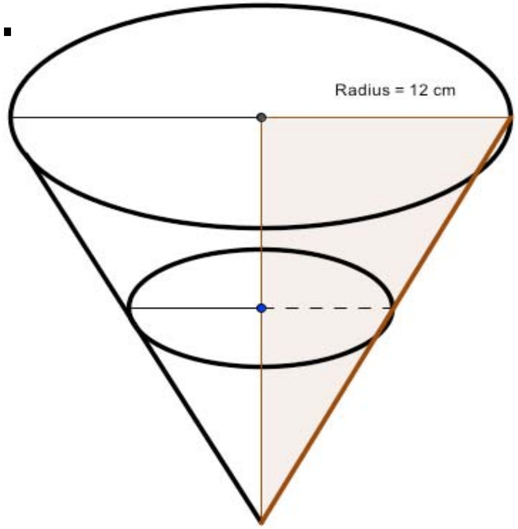
Click for suggestions



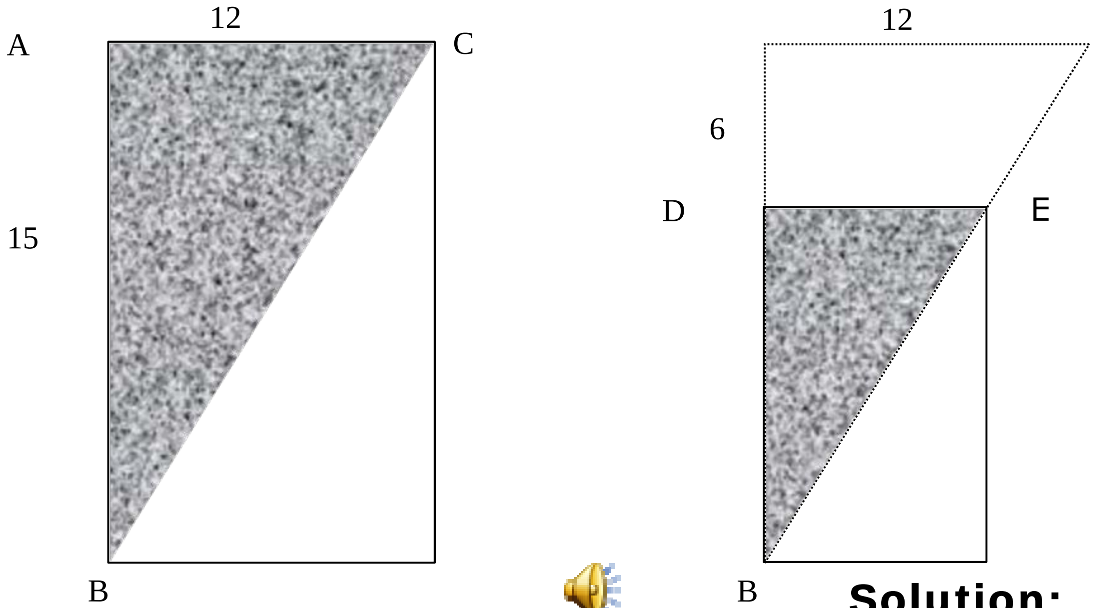
The task:

A cone-shaped hourglass drips sand from the bottom point. When it is filled up with sand, the radius of the base is 12 cm and the height is 15 cm. After the sand drops for 12 seconds, the height of the sand decreases by 6 cm. Find the radius of the base at this time. Share your solution process with a colleague.

Suggestion:
Note the triangle shaded in brown. Is there another that is similar to it that involves the smaller radius? How could you use the height measurements?



Hour Glass Solution



Solution:
Since DB is 6 cm shorter than AB, DB = 9 cm.

Because ΔABC and ΔDBE are similar to each other,

$$\frac{DE}{9} = \frac{12}{15} \Rightarrow DE = \frac{108}{15} = 7.2$$

Comments on Complexity:

- Like the shadow problem, two similar triangles were drawn on the picture.
- The correspondences are not necessarily camouflaged, but are difficult to locate within the other components of the representation.
- There are some verbal cues suggesting correspondence such as the labels *radius* and *smaller radius*.

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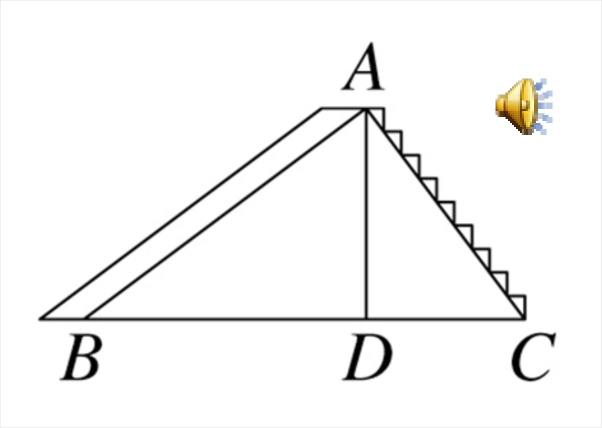
Water Slide Task–I

Michigan Adventure has a water slide. The length of the stairs (AC) is 20 meters. The height of the slide (AD) is 16 meters. If the angle formed by sides AB and AC is 90 degrees, how wide is the water slide BC?

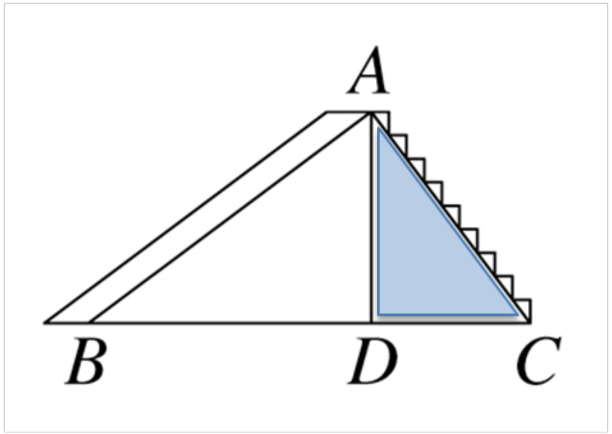


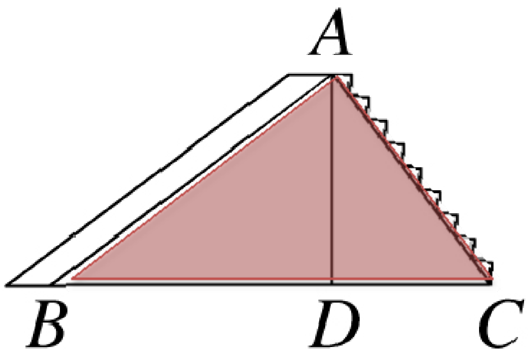
Work with a few colleagues to find the solution. Share your solution process with the whole group.

Click for suggestion

Water Slide Task–II

Michigan Adventure has a water slide. The length of the stairs (AC) is 20 meters. The height of the slide (AD) is 16 meters. If the angle formed by sides AB and AC is 90 degrees, how wide is the water slide BC?





The blue and red triangles are similar because they have two angles in common. (Can you tell which?)

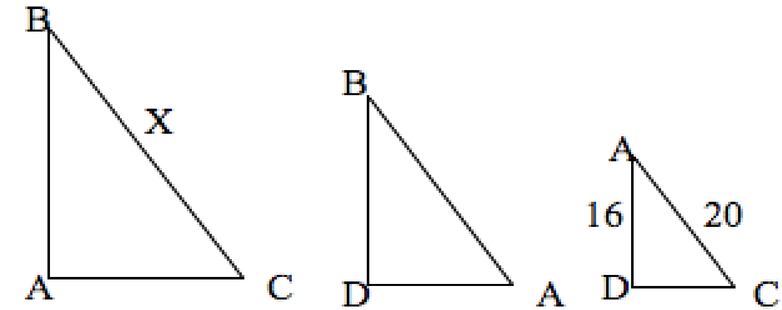
Work with a few colleagues to find the solution.
Discuss your solution process with the whole group.

Solution:

Comments on Complexity:

- The similar triangles were included in the diagram, but were not the only triangles drawn.
- The correspondences are camouflaged by rotation.
- Furthermore, the side AC is used in both of the triangles to be found. However, the two uses are not corresponding sides.
- There are no verbal or contextual cues to highlight potential correspondences—the only way to detect them is through an assumption or proof that the two triangles are similar.

Recognize ΔBAC and ΔACD and ΔBAD are similar.



There are several ways this problem can be solved once we recognize the above relationship. One possible solution follows:

$$\frac{AC}{AD} = \frac{BC}{BA}$$

Let BC= x, then $\frac{20}{16} = \frac{x}{BA}$. Therefore, BA= $\frac{4}{5}x$

Apply the Pythagorean theorem to ΔBAC.

Solve the following equation:

$$20^2 + (\frac{4}{5}x)^2 = x^2$$

$$x = \frac{100}{3}$$

Interventions: Mathematics Teaching in the Middle School Article

- Snow and Porter (Mathematics Teaching in the Middle School, February 2009) have written an article about ratios and proportions that could be utilized to further your students’ understanding of indirect measurement.

- The article, “Ratios and Proportions: They Are Not All Greek to Me,” includes a history of early uses of ratios and proportions in Greece, an analysis of early attempts at a theory concerning ratios and proportions, geometric applications, and applications to numbers. An activity sheet is included which contains indirect measurement problems.

To Review:

- Similar triangles can be used to measure distances that are difficult to measure in real life such as the height of buildings, distances across bodies of water, or even diagonals of objects.

- In order to apply proportional reasoning to do indirect measurement, students must be aware of the proportional relationship within the situation as well as recognize key correspondences. This is difficult to do if camouflage exists and contextual or verbal cues are missing. **Even students who are able to reason proportionally about these situations may not recognize them as opportunities to use this type of reasoning.**

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Lesson 2:
Candle–Burning Tasks

In this session, you will develop the ability to:

- identify a direct, inverse, or non-proportional relationship that exists in a given situation
- describe a direct, inverse, or non-proportional relationship with table, graph or equation

You will work your way through five word problems. Each is set in the context of candle-burning[1].

[1] Lim, K. H. (2009). Burning the candle at just one end: Using non-proportional examples helps students determine when proportional strategies apply. *Mathematics Teaching in the Middle School*, *14* (8), 492–500.

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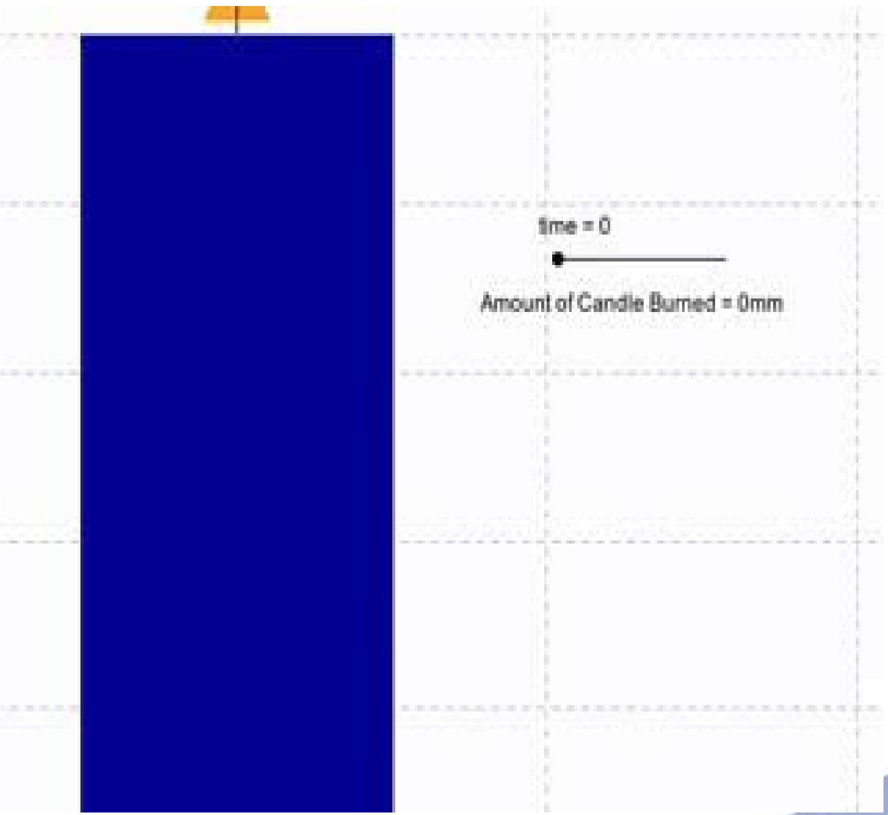
#*1*: A candle is burning at a constant rate.
It has burned 12 mm after 20 minutes.

a) How many millimeters has the candle burned after 50 minutes?

b) Let ***b*** represent the number of millimeters the candle has burned after ***t*** minutes. Write an equation to relate ***b*** and ***t***.

c) Sketch the graph of ***b*** in terms of ***t***.



#2: A candle is burning at a constant rate. When it has burned 30 mm, its height is 75 mm.

a) When it has burned 60 mm, what will the candle's height be?

b) Let ***h*** represent the candle's height when it has burned ***x*** mm. Write an equation to relate ***h*** and ***x***.

c) Sketch the graph of ***h*** in terms of ***x***.

Click for Suggestions

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#2: A candle is burning at a constant rate. When it has burned 30 mm, its height is 75mm.

a) When it has burned 60 mm, what will the candle's height be?

b) Let ***h*** represent the candle's height when it has burned ***x*** mm. Write an equation to relate ***h*** and ***x***.

c) Sketch the graph of ***h*** in terms of ***x***.

- Is there extraneous information in the problem?
- There is a constant relationship in this problem; what is it?

#3: An altar always needs to be lit using one special candle at a time.

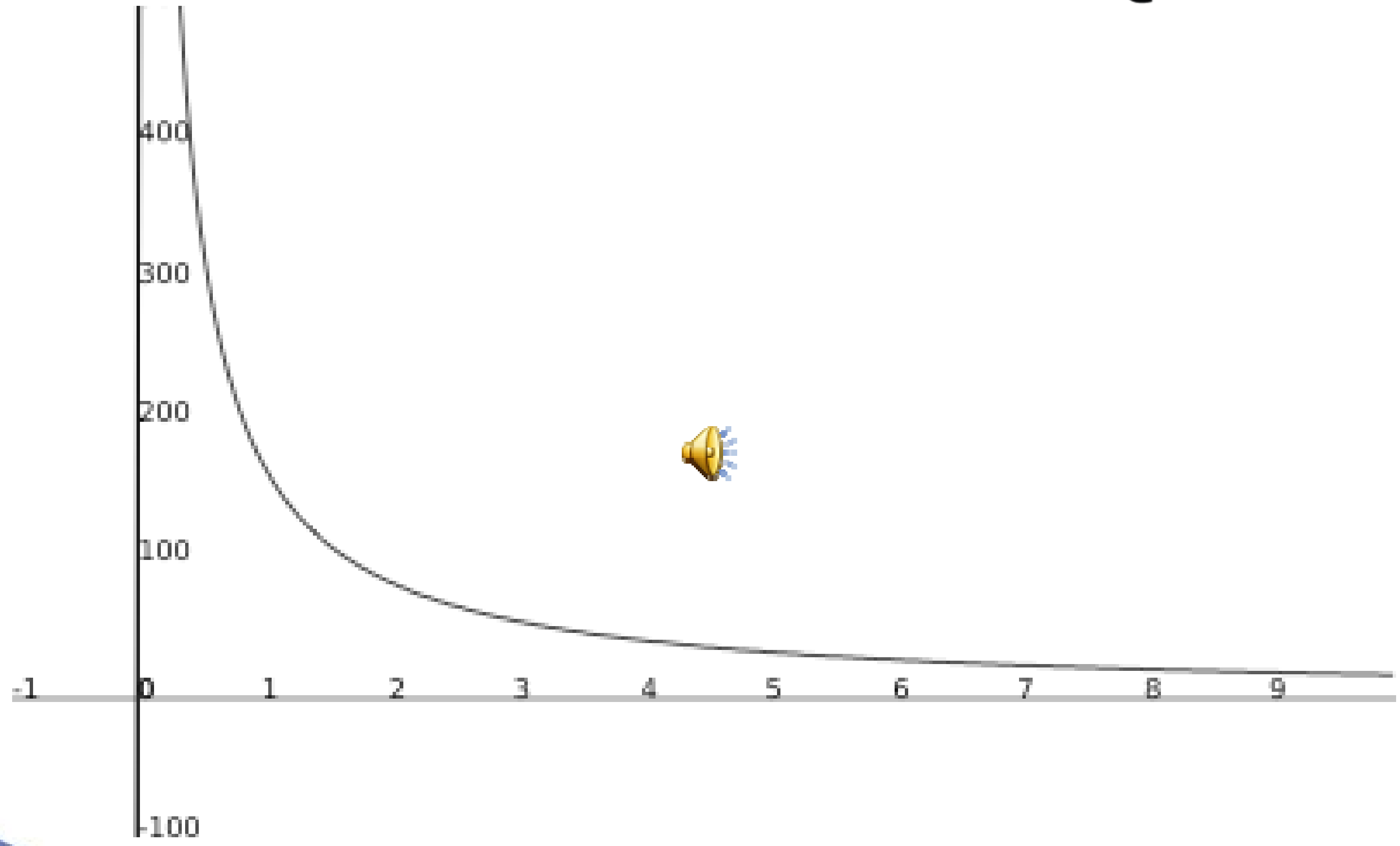
Using candles that last 7 hours each, 24 such candles would be needed.

a) If the church uses different candles that last 8 hours each, how many 8-hour candles will the church need?

b) Let ***n*** represent the number of candles that the church needs, with each candle lasting ***t*** hours. Write an equation to relate ***n*** and ***t***.

c) Sketch the graph of ***n*** in terms of ***t***.

The graph of $n = \frac{168}{t}$



Looking Back (I)

Now, compare your solutions for the first three problems to those of your partner(s).

Task one: A candle is burning at a constant rate. It has burned 12 mm after 20 minutes.

Task two: A candle is burning at a constant rate. When it has burned 30 mm, its height is 75mm.

Task three: How many special altar candles are needed?

Together, decide which of the three is a direct proportion, inverse proportion, or non-proportional situation. What is the basis for your decisions?

Direct Proportion	Inverse Proportion	Non-Proportion

Looking Back (II)

Task one: A candle is burning at a constant rate. It has burned 12 mm after 20 minutes.

Task two: A candle is burning at a constant rate. When it has burned 30 mm, its height is 75mm.

Task three: How many special altar candles are needed?

Direct Proportion	Inverse Proportion	Non-Proportion
Task One There is a constant rate between two quantities, that is *the length of burned candle/ the amount of time = b/t = 12/20 = 0.6* Therefore, the relationship between *b* and *t* can be written with the following equation: $b = 0.6t$ The graph is a straight line passing (0,0) with slope equal to 0.6.	**Task Three** A constant exists: the number of candle-hours for a week. This number equals 7 x 24 = 168. This implies (# candles) × (hours candle lasts) = $nt = 168$ Therefore, the relationship between *n* and *t* can be written with the following equation: $n = 168/t$ What does the graph of $n = 168/t$ look like?	**Task Two** A constant exists, but it's a constant sum. (amount of burned candle) + (amount of remaining candle) = original length which is a constant. Therefore, $h + x = 30 + 75 = 105$ and $h = 105 – x$ The graph of h in terms of x is a decreasing line passing (105, 0) with slope = -1.

Making Conjectures

- What can you say about the equation of a direct proportional relationship? How about that of the inverse proportion?

- What can you say about the graph of a direct proportional relationship? How about that of the inverse proportion?

- Share your responses with the whole group.

More Candle-Burning Tasks

Next, you will be working on two more candle-burning tasks. One of them is similar to one of the first three tasks, and the other one has a different type of relationship embedded in the problem. Work through them carefully to see if you can tell them apart.

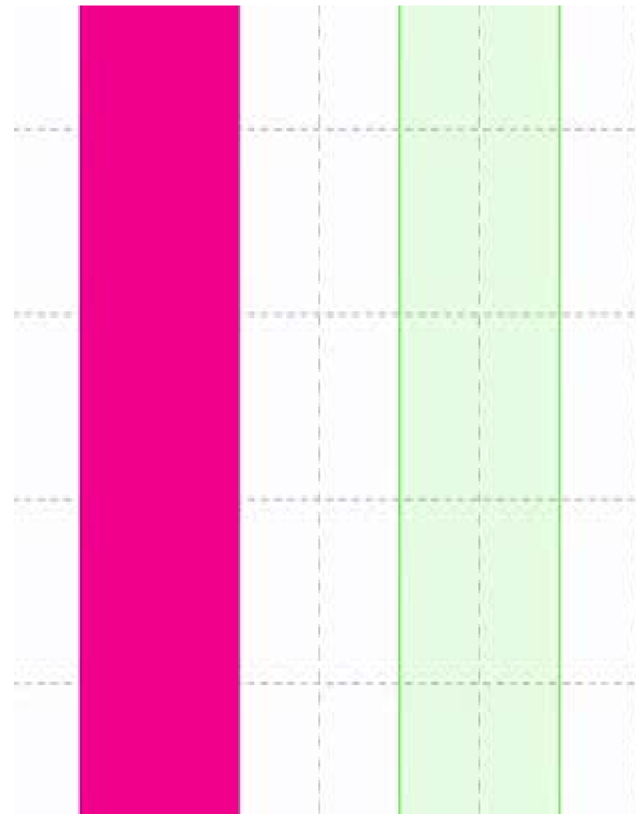
#*4:* Two identical candles, A and B, lit at different times, are burning at the same constant rate.

When candle A has burned 20 mm, candle B has burned 12 mm.

a) When candle B has burned 30 mm, how many millimeters will candle A have burned?

b) Let ***a*** represent the number of millimeters that candle A has burned when candle B has burned ***b*** mm. Write an equation to relate ***a*** and ***b***.

c) Sketch the graph of ***a*** in terms of ***b***.



Click image
to animate candles.

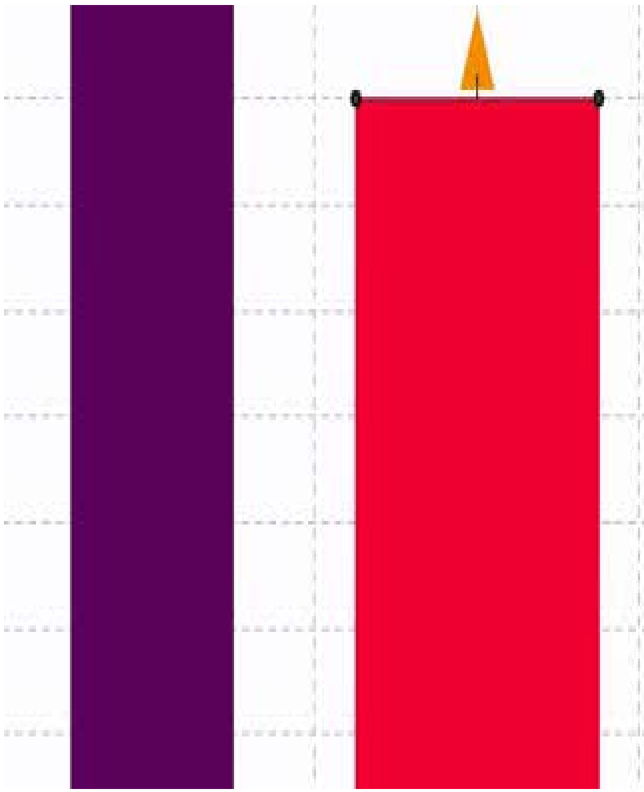
#5: Two different candles, P and Q, lit at the same time, are burning at different, but constant rates.

When candle P has burned 16 mm, candle Q has burned 10 mm.

a) When candle Q has burned 35 mm, how many millimeters will candle P have burned?

b) Let ***p*** represent the number of millimeters that candle P has burned when candle Q has burned ***q*** mm. Write an equation to relate ***p*** and ***q***.

c) Sketch the graph of ***p*** in terms of ***q***.



Click image
to animate candles.

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Looking Back (III)

Now, compare your solutions for these two tasks to those of your partner(s).

Task Four: Two identical candles, A and B, lit at different times, are burning at the same constant rate.

Task Five: Two different candles, P and Q, lit at the same time, are burning at different, but constant rates.

Together, decide which of the two is similar to one of the first three problems? Which one has a new type of relationship. What is the basis for your decisions?

Direct Proportion	Inverse Proportion	NonProportion / Constant Sum	New Relationship
•Task One	•Task Three	•Task Two	

Looking Back (IV)

Direct Proportion	Inverse Proportion	Non-Proportion: Constant Sum	New Relationship
•**Task One** •**Task Five** There is a constant rate between *length of burned Candle P/ length of burned Q candle* $= \boldsymbol{p/q} = 16/10 = 1.6$ Therefore, the relationship between ***p*** and ***q*** can be written with the following equation: $\boldsymbol{p} = 1.6\boldsymbol{q}$ The graph is a straight line passing (0,0) with slope equal to 1.6.	•**Task Three**	•**Task Two**	**Task Four** (Length of burned candle A) – (Length of burned candle B) = a constant Therefore, a - b = 20 - 12 = 8 and a = 8 + b The graph of a in terms of b is an increasing line passing (0, 8) with slope equal to 1.

Reconsider Tasks #2 and #4 (I)

Both Tasks #2 and #4 contain a constant term, just like those in the direct proportion and inverse proportion problems. However, instead of having a constant rate (can be written in the quotient form) or a constant product, #2 contains a constant sum and #4 contains a constant difference.

Reconsider Tasks #2 and #4 (II)

It is easy to mistake Tasks #2 and #4 as problems with proportional relationships because they read like a typical "missing value" task. These tasks typically ask students to find a fourth value with three given values much like the indirect measurement problems. It is difficult to discern when it is appropriate to use proportional reasoning and when it is not.

Therefore, it is important to provide students opportunity to think through these problems to deepen their understanding of proportional relationships.

Reconsider Tasks #2 and #4 (III)

Task #4 is also challenging because there are several quantities embedded in this problem:

- The length of the candle burned (L_A&L_B)
- The time to burn that amount (T_A&T_B)
- The rate of burning= $L_A/T_A=L_B/T_B$

Since the rate of burning is the same, the length of the candle burned is also the same for both A and B for a given amount of time.

Interventions:
Insights Into Algebra I: Teaching for Learning

- The Annenberg Foundation has produced an eight-part Web-based professional development workshop to assist mathematics instructors in algebraic pedagogy.
- Workshop 7: Direct and Inverse Variation provides more rich real-life applications in these areas.
- Refer to the following Web-site:

http://www.learner.org/workshops/algebra/workshop7/lessonplan1d.html

To review:

There are distinct mathematical characteristics of proportional situations.

Algebraically,

1) If there is a direct proportional relationship between quantities x and y, then there exists a constant k such that $y = kx$.
2) If there is an inverse proportional relationship between quantities x and y, then there exists a constant k such that $x \cdot y = k$.

Beware!

Quantities x and y do not relate to each other proportionally if $x + y = k$ or $x - y = k$.

Graphically,

1) If there is a direct proportional relationship between *x* and *y*, then the graph of y = f(x) is a straight line passing through (0,0).

2) If there is an inverse proportional relationship between *x* and *y*, then the graph of y = f(x) is non-linear.

Beware!

Any line that does not go through (0,0) with a positive slope does not imply a proportional relationship.

Final Challenges

Solve the problems on the next slide with your partners. Determine which of these are direct proportional, inverse proportional, or non-proportional situations.

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1. 5 U.S. dollars can exchange for 160 Taiwanese dollars. How many U.S. dollars can exchange for 576 Taiwanese dollars?
2. When the temperature reads 68° F, it is also 20° C. When the temperature reads 59 ° F, how many degrees Celsius will this be?
3. If Ed can paint the bedroom by himself in 12 hours, and his friends, Jake and Juan, work at the same pace as Ed does. How long will it take to paint the room if all three boys work together?
4. Bob and Marty like to run laps together because they run at the same pace. Today, Marty started running before Bob came out of the locker room. Marty had run 7 laps by the time that Bob ran 3. How many laps had Marty run by the time that Bob had run 12?

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Answers to the Final Challenges

1.	Direct proportion	Answer: 18 U.S. dollars
1.	Non-proportion	Answer: 15 degrees
2.	Inverse proportion	Answer: 4 hours
4.	Non-proportion	Answer: 16 laps

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Module Reflection

If you are now teaching proportional reasoning or have in the past, think about any strategies you have previously used for teaching indirect measurement and direct and inverse proportions. Compare and contrast your strategies with those discussed in this session.

Many middle school teachers have their own favorite application problems for teaching proportional reasoning using indirect measurement and direct and inverse proportions effectively. Share yours with others at this time.

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Evaluation

Write a statement or two that summarizes what you have learned during this session.

What would you like to have added to this second session on the teaching and learning of indirect measurement and direct and inverse proportion.

E-mail your responses for the above to
ruth.a.meyer@wmich.edu

Appendix

- Facilitator Notes
  Proportional Reasoning - II - Facilitator Notes.docx

- Final Challenge Questions and Answer Key
  Final Challenge Problems and Answer Key.doc

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