Fluid Flow and Optical Flow

— Determination of Velocity Field from Images

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Objective

To develop a general theoretical framework for extracting velocity fields from various flow visualization images

Key Issues:

- Projected Motion Equation & Optical Flow Equation
- Variational Formulation & Euler-Lagrange Equations
- Numerical Algorithms
- Application to Various Flow Visualizations
Particle Image Velocimetry (PIV)

Images of Discrete Particles

Cross-Correlation

Velocity Vectors
Extracting Velocity Fields from Images of Continuous Patterns?
Optical Flow: Visual Motion in Images

*Image Intensity Constraint Equation in Computer Vision:* 
\[
\frac{\partial I}{\partial t} + u \cdot \nabla_x I = 0
\]

**Criticisms for Optical Flow:**
- *not related to any physical process*
- *does not have any physical meaning*

**The important question in flow measurements:**

*Link between Fluid Flow and Optical Flow*
Projected Motion Equations
— Modeling of Imaging through Fluid Flows

Projection onto Image:
- Geometrical (Perspective)
- Radiometric

Transport Equations in Flows

Projection

Projected Motion Equations
**Projective Projection Transformation**

**Collinearity Equations:**

\[ x^I - x_p^I + \delta x^I = -c \begin{pmatrix} m_1^T (X - X_c) \\ m_3^T (X - X_c) \end{pmatrix} \]

\[ x^2 - x_p^2 + \delta x^2 = -c \begin{pmatrix} m_2^T (X - X_c) \\ m_3^T (X - X_c) \end{pmatrix} \]

**Camera Parameters:**

\[ \Pi = (\omega, \phi, \kappa, X_c^1, X_c^2, X_c^3, c, x_p^1, x_p^2, K_1, K_2, P_1, P_2) \]
Radiometric Projection onto Image

Radiance projected onto a plane parallel to the image plane

\[ I(\mathbf{x}, t) = c \int L(\mathbf{X}_1, \mathbf{X}_2, t; \theta, \psi) \cos \theta \, d\Omega \]

solid angle:
\[ d\Omega = \sin \theta \, d\theta \, d\psi \]

For a camera that is sufficiently away from an observed body

\[ I(\mathbf{x}, t) = c \, L(\mathbf{X}_1, \mathbf{X}_2, t; \theta_c, \psi_c) \cos \theta_c \, \Delta \Omega \]
Case 1: *Laser-Induced Fluorescence*

Fluorescent radiance across a laser sheet:

\[ L( \bar{X}^1, \bar{X}^2, t ) = c_\psi L_0( \bar{X}^1, \bar{X}^2 ) \int_{\Gamma_1}^{\Gamma_2} \psi( \bar{X}, t ) d \bar{X}^3 \]

Transport equation for a passive scalar

\[ \frac{\partial \psi}{\partial t} + \nabla \cdot (\psi U) = D_\psi \nabla^2 \psi \]

Projected Motion Equation in Incompressible Flow

\[ \frac{\partial g}{\partial t} + \nabla_{12} \cdot \left( g \left\langle U_{12} \right\rangle_\psi \right) = D \nabla^2_{12} g + D c B + c n \cdot \left( \psi U \right) \bigg|_{\Gamma_2}^{\Gamma_1} \]

where

\[ g = \frac{L}{L_0} \]

\[ \left\langle U_{12} \right\rangle_\psi = \frac{\int_{\Gamma_1}^{\Gamma_2} \psi U_{12} d \bar{X}^3}{\int_{\Gamma_1}^{\Gamma_2} \psi d \bar{X}^3} \]

Path-averaged velocity
Case 2: *Transmittance Image from Scalar*

Radiance through a scalar to a camera:

\[
L( X_1, X_2, t ) = L_0( X_1, X_2 ) \exp \left( - \int_{\Gamma_1}^{\Gamma_2} \beta( X, t ) dX_3 \right)
\]

**Projected Motion Equation**

\[
\frac{\partial g}{\partial t} + \nabla_{12} \cdot \left( g \langle U_{12} \rangle_{\psi} \right) = D \nabla_{12}^2 g - D \varepsilon B - \varepsilon n \cdot \left( \psi U \right) \bigg|_{\Gamma_1}^{\Gamma_2}
\]

where \( g = \ln( L / L_0 ) \)
Case 3: *Images of Density-Varying Flows*

**Image intensity in Schlieren images:**

\[
\frac{I - I_K}{I_K} = C \int_{\Gamma_1}^{\Gamma_2} \frac{\partial \rho}{\partial X_2} dX_3
\]

**Continuity equation:**

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho U) = 0
\]

**Projected Motion Equation**

\[
\frac{\partial g}{\partial t} + \nabla_{12} \cdot \left( g \langle U_{12} \rangle_\rho \right) = 0
\]

where

\[
g = \int_{X_{20}}^{X_2} (I / I_K - 1) dX_2 \quad \langle U_{12} \rangle_\rho = \frac{\int_{\Gamma_1}^{\Gamma_2} \rho U_{12} dX_3}{\int_{\Gamma_1}^{\Gamma_2} \rho dX_3}
\]
Case 3: Images of Density-Varying Flows

Image intensity in *shadowgraph* images:

\[
\frac{I - I_T}{I_T} = C \int_{\Gamma_1}^{\Gamma_2} \nabla_{12}^2 \rho \, dX_3
\]

**Projected Motion Equation**

\[
\frac{\partial g}{\partial t} + \nabla_{12} \cdot \left( g \langle U_{12} \rangle \rho \right) = 0
\]

where \( \nabla_{12}^2 g = I / I_T - 1 \)

For *transmittance* images:

\[
\frac{\partial g}{\partial t} + \nabla_{12} \cdot \left( g \langle U_{12} \rangle \rho \right) = 0
\]

where \( g = I / I_T \)
**Case 4: Transmittance Image through Scattering Particulate Flow**

**Radiance through a scattering particulate flow:**

\[
L(\tau) = L_0 \exp(-\tau) + \int_0^\tau S^{(\infty)}(\tau', s) \exp[-(\tau - \tau')] \, d\tau'
\]

**Disperse phase number equation:**

\[
\frac{\partial N}{\partial t} + \nabla \cdot (N \mathbf{U}) = 0
\]

**Projected Motion Equation**

\[
\frac{\partial g}{\partial t} + \nabla_{12} \cdot \left( g \langle U_{12} \rangle_N \right) = w_{wx} \left\langle C_{ext} \right\rangle_a \mathbf{n} \cdot (N \mathbf{U})_{\Gamma_2}^{\Gamma_1}
\]

where

\[
g = \left[ L(\tau_H) - L_0 \right] / \left[ S^{(\infty)}(0) - L_0 \right] \quad \langle U_{12} \rangle_N = \frac{\int_{\Gamma_1}^{\Gamma_2} N U_{12} \, dX_3}{\int_{\Gamma_1}^{\Gamma_2} N \, dX_3}
\]
Case 5: Scattering Image toward Incident Direction from Particulate Flow

Radiance scattered back from particulate flow:

\[ L^- (\tau_H) = L^- (0) \exp(-\tau_H) + \int_0^{\tau_H} Q_{sca}(\tau_H - \tau') \exp\left[-(\tau_H - \tau')\right] d\tau' \]

Projected Motion Equation

\[ \frac{\partial g}{\partial t} + \nabla_{12} \cdot \left( g \mathbf{\langle U_{12} \rangle}_N \right) = w_{wext} \mathbf{\langle C_{ext} \rangle}_a \mathbf{n} \cdot (NU) \bigg|_{\Gamma_2}^{\Gamma_1} \]

where

\[ g = \frac{[L^- (\tau_H) - L^- (0)] / [Q_{sca}(0) - L^- (0)]}{\int_{\Gamma_1}^{\Gamma_2} N \mathbf{U}_{12} dX_3} \]

\[ \mathbf{\langle U_{12} \rangle}_N = \frac{\int_{\Gamma_1}^{\Gamma_2} N dX_3}{\int_{\Gamma_1}^{\Gamma_2} N dX_3} \]
Case 6: **Laser-Sheet-Illuminated Particle Images**

Particle scattering radiance across a laser sheet:

\[
L(\bar{X}^1, \bar{X}^2, t) = C_{sca} L_0(\bar{X}^1, \bar{X}^2) \int_{\Gamma_1}^{\Gamma_2} N_p(\bar{X}, t) \, d\bar{X}^3
\]

Projected Motion Equation for PIV

\[
\frac{\partial (L/L_0)}{\partial t} + \nabla_{12} \cdot \left[ (L/L_0) \langle U_{p12} \rangle_{N_p} \right] = C_{sca} N \cdot (N_p U_p)_{\Gamma_1}
\]

Particle accumulation in laser sheet

Path-averaged particle velocity:

\[
\langle U_{p12} \rangle_{N_p} = \frac{\int_{\Gamma_1}^{\Gamma_2} N_p U_{p12} \, d\bar{X}^3}{\int_{\Gamma_1}^{\Gamma_2} N_p \, d\bar{X}^3}
\]
Particle Velocity & Path-Averaged Velocity

**Discrete particles image:**

\[ L / L_0 = \sum_{i=1}^{M} g_i \]

where

\[ g_i = \frac{1}{2 \pi \sigma_i^2} \exp \left[ - \left( \frac{(X^1 - X^1_{pi})^2 + (X^2 - X^2_{pi})^2}{2 \sigma_i^2} \right) \right] \]

**When** \( \max(\sigma_i^2) \rightarrow 0 \) \( g_i \rightarrow \delta( X^1 - X^1_{pi}, X^2 - X^2_{pi} ) \)

**Relation between Discrete and Continuous Forms:**

\[ \left( \frac{dX^1_{pi}}{dt}, \frac{dX^2_{pi}}{dt} \right)^T = \delta( X^1 - X^1_{pi}, X^2 - X^2_{pi} ) \left\langle U_{p12} \right\rangle_{N_p} \]
Generic Form of Projected Motion Equation

Flow Visualizations:

(1) Transmittance images of passive scalar
(2) Schlieren images, Shadowgraphs, Transmittance images of density-varying flows
(3) Laser-induced fluorescence (incompressible & compressible)
(4) Transmittance and scattering images for particulate flows
(5) Laser-sheet-illuminated particle image (PIV)

\[
\frac{\partial g}{\partial t} + \nabla_{12} \cdot \left( g \langle U_{12} \rangle_\psi \right) = D_\psi \nabla_{12}^2 g + B(\Gamma_1, \Gamma_2)
\]

Path-averaged velocity:

\[
\langle U_{12} \rangle_\psi = \frac{\int_{\Gamma_1}^{\Gamma_2} \psi U_{12} \, d\bar{X}^3}{\int_{\Gamma_1}^{\Gamma_2} \psi \, d\bar{X}^3}
\]
<table>
<thead>
<tr>
<th>Visualization Technique</th>
<th>Form of $g$</th>
<th>Definition of $\psi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Laser-Sheet-Induced Fluorescence Image</td>
<td>$g = L / L_0$</td>
<td>Scalar density $\psi$</td>
</tr>
<tr>
<td>Transmittance Image through Passive Scalar</td>
<td>$g = \ln(L / L_0)$ or $g = -\Delta L / L_0$</td>
<td>Scalar density $\psi$</td>
</tr>
<tr>
<td>Schlieren Image</td>
<td>$g = \int_{x_{x_0}}^{x_f} (I / I_K - 1) dX_2$</td>
<td>Air density $\rho$</td>
</tr>
<tr>
<td>Shadowgraph Image</td>
<td>$\nabla_{i2}^2 g = I / I_T - 1$</td>
<td>Air density $\rho$</td>
</tr>
<tr>
<td>Transmittance Image</td>
<td>$g = I / I_T$</td>
<td>Air density $\rho$</td>
</tr>
<tr>
<td>Transmittance Image through Scattering Particulate Flow</td>
<td>$g = [L(t_H) - L_0] / [S^{(\infty)}(0) - L_0]$</td>
<td>Particle density $N$</td>
</tr>
<tr>
<td>Scattering Image toward Incident Direction from Particulate Flow</td>
<td>$g = [L^-(t_H) - L^-(0)] / [Q_{sca}(0) - L^-(0)]$</td>
<td>Particle density $N$</td>
</tr>
<tr>
<td>Laser-Sheet-Illuminated Particle Image</td>
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</tr>
</tbody>
</table>
Physics-Based Optical Flow Equation

When the projected motion equation is expressed in the image coordinates, we obtain a *physics-based optical flow equation*

$$\frac{\partial g}{\partial t} + \nabla \cdot (g \mathbf{u}) = f(\mathbf{x}^1, \mathbf{x}^2, g)$$

Optical flow has a clear physical meaning:

$$\mathbf{u} = (u_1, u_2) = \lambda \langle \mathbf{U}_{12} \rangle_\psi$$

Diffusion and boundary terms:

$$f(\mathbf{x}^1, \mathbf{x}^2, g) = \lambda^2 D_\psi \nabla^2 g + B(\Gamma_1, \Gamma_2)$$
Variational Formulation

Functional:

\[ J(u) = \int_A \left( \partial g / \partial t + \nabla \cdot ( g u ) - f \right)^2 \, dx_1dx_2 + \alpha \int_A \left( |\nabla u_1|^2 + |\nabla u_2|^2 \right) \, dx_1dx_2 \]

Euler-Lagrange Equation:

\[ g \nabla \left[ \partial g / \partial t + \nabla \cdot ( g u ) - f \right] + \alpha \nabla^2 u = 0 \]

Neumann Boundary Condition:

\[ \partial u / \partial n = 0 \quad \text{on} \quad \partial \Omega \]

Numerical solution: Finite difference & Jacob iteration
Error Analysis

Error propagation equation:

\[ g \nabla \left[ D_g + \nabla g \cdot (\Delta u) \right] + \alpha \nabla^2 (\Delta u) = \alpha \left( \frac{\Delta g}{g} \right) \nabla^2 u \]

Elemental error terms:

\[ D_g = \Delta \left( \frac{\partial g}{\partial t} \right) + \Delta (\nabla g) \cdot u + g \Delta (\nabla \cdot u) + \Delta g (\nabla \cdot u) \]

Error Constraint:

\[ \left\| \nabla g \right\|_{\text{char}}^{-1} \Delta t = \varepsilon \]

For a given \( \Delta t \) the intensity gradient: \( \left\| \nabla g \right\|_{\text{char}} \)

Optical flow error: \( \Delta u \)
Simulations: Grid Images

Oseen vortex pair and uniform flow

$t = 0 \text{ s}$

$t = 0.02 \text{ s}$
Simulations: Grid Images

Extracted velocity vectors and streamlines
Comparison with Exact Profile

- Exact Profile across Two Vortex Cores
- Present Method
- Horn-Schunck Estimator
Optical Flow Error Map
Grid Images with Different Intensity Gradients
Effect of Intensity Gradient on Optical Flow Error

![Graph showing the effect of intensity gradient on optical flow error. The x-axis represents the intensity gradient (counts/pixel), and the y-axis represents the error in optical flow (pixels/s). The graph compares different methods, including the Present Method and the Horn-Schunck Estimator, across varying intensity gradients.]
Effect of Time Interval on Optical Flow Error

![Graph showing the effect of time interval on optical flow error](image-url)
Optical Flow Error as a Function of $\left\| \nabla g \right\|_{char}^{-1} \Delta t$
Optical Flow vs Cross-Correlation for PIV Images

- **Optical Flow — Differential Approach**
- **Correlation — Integral Approach**

*PIV images as a non-smooth random noise field are not good for optical flow*

*Oseen-vortex pair in uniform flow*
Error Analysis and Relevant Parameters

**Error Estimate**

\[
\varepsilon = \frac{\Delta x_p}{\sqrt{\frac{c_1 d_p^2}{\|u_p\|^2} + \frac{c_2}{\|u_p\|^2} + \frac{c_3}{d_p^2} + \frac{c_4}{N_p^m} \frac{\|\Delta^m x_p\|^2}{\|\Delta x_p\|^2}}}
\]

**Four Error Parameters**

- **Particle displacement** \(\|\Delta x_p\|\)
- **Particle diameter** \(d_p\)
- **Particle velocity gradient** \(\|\nabla u_p\|\)
- **Particle image density** \(N_p\)
Optical Flow vs Correlation in Parameter Space

- **Max Displacement vs RMS Error**: The plots show the relationship between max displacement and RMS error for different parameter combinations.
- **Velocity Gradient vs RMS Error**: The plots illustrate the impact of velocity gradient on RMS error.
- **Particle Image Density vs RMS Error**: The plots demonstrate how particle image density affects RMS error.
- **Particle Image Diameter vs RMS Error**: The plots analyze the relationship between particle image diameter and RMS error.
Simulations: Low-Density Particle Images

Oseen vortex pair and uniform flow

$t = 0 \text{ s}$

$t = 0.02 \text{ s}$
Comparison with PIV

Displacement < 1 pixel

Good for optical flow
Not good for PIV

Optical flow method
with Gaussian filtering

PIV
Simulations: High-Density Particle Images

Optical flow method with Gaussian filtering
Strongly Excited Turbulent Jet

Tube Diameter: 9.5 mm
Excitation Frequency: 15 Hz
Laser Pulse Interval: 10 μs
Velocity Fields in a Cycle (Optical Flow Method)

(a) 0T, (b) T/8, (c) T/4, (d) T/2, (b) 3T/4, where T = 1/15 s.
(a) 0T, (b) T/8, (c) T/4, (d) T/2, (b) 3T/4, where T = 1/15 s.
Optical Flow Method vs. TSI Insight5
Optical Flow Method vs. TSI Insight5

- **Graph 1:**
  - Title: x/D = 1.04, y/D = 0
  - Legend: Optical Flow Method, Correlation Method (TSI)
  - x-axis: t/T
  - y-axis: u (m/s)

- **Graph 2:**
  - Title: x/D = 1.04, y/D = -0.46
  - Legend: Optical Flow Method, Correlation Method (TSI)
  - x-axis: t/T
  - y-axis: u (m/s)
Snapshot Field in Impingement Region of Normal Impinging Jet

Optical Flow

PIV Image

Correlation (LaVision)
Impinging Jet: Impingement Region

Comparison between Snapshot Velocity Profiles

**X-velocity component** | **Y-velocity component**

---

[Graph showing comparison between snapshot velocity profiles for X and Y components.]
Impinging Jet: Wall-Jet Region

Comparison between Snapshot Velocity Profiles

X-velocity component  Y-velocity component
Ensemble-Averaged Fields in Wall-Jet Region of Normal Impinging Jet

**Optical Flow**

- Kinetic Energy (pixels/unit time)$^2$

**Correlation**

- Kinetic Energy (pixels/unit time)$^2$

**Turbulent Kinetic Energy**

- Reynolds Stress (pixels/unit time)$^2$

**Reynolds Stress**
Impinging Jet: Wall-Jet Region

Comparison between Ensemble-Averaged Profiles

Turbulent Kinetic Energy  Reynolds Stress
Flow Structures of Jupiter’s Great Red Spot

*Galileo 1996 Images of the GRS (G1)*

$t = 0 \text{ s}$  
$t = 4320 \text{ s}$
Global Structures of the GRS

**Velocity Vectors (resolution reduced by 4)**

**Relative Vorticity**

- **High-speed, near-elliptical, anti-cyclonical collar**
- **Low-speed inner region**
Zonal Velocity Profile across the GRS along the Minor Axis

Cyclonic, counter-rotational motion near the center

It will be pointed out that this structure is intrinsic.
Cyclonic Motion & Outward Spiraling Source Node

Topological Constraint:

\[ \# N - \# S = 1 \]

Consequence:

One long-lived node necessary for the existence of the GRS
Cyclonic Source Node & Convention Instability

In a Lagrangian sense by following a fluid parcel along $z$ to a source node,

$$- \frac{\partial w}{\partial z} \approx \int_{0}^{t} N^2 \left[ z( t' ) \right] dt' > 0$$

Convection Instability

where

$$N^2 = \left( \frac{g}{T} \right) \left( \frac{\partial T}{\partial z} + \frac{g}{c_p} \right)$$

(N is the buoyancy frequency)

The surface vorticity at a source node:

$$\omega_s \approx f_o^2 N_s^{-2} \left( -\frac{\partial w}{\partial z} \right) > 0$$

Cyclonic Rotation

The relative vorticity is intensified by the convection-induced stretching of the planetary vorticity associated with the Coriolis force.
Extracting High-Resolution Velocity Fields on a Full-Scale Car Based on Smoke Visualization

Optical Flow Diagnostics: Large regions at one vector per pixel

Small diagnostic regions: Problem in full-scale wind tunnels
Flow over Car in Full-Scale Wind Tunnel

Velocity Vectors Extracted by Using Optical Flow Method
Unique Solution in Bounded Variation (BV) Space

**Theorem 3.1:**

The minimization problem \( \inf_{u \in \text{BV}(\Omega)} J(u) \) admits a unique solution in BV space.

**The Convergence Theorem**

**Theorem 4.5:**

Let \( (u^0, a^0, b^0) \in W^{2,2}(\Omega) \times L^2(\Omega) \times L^2(\Omega) \) be given. Then the sequence \( (u^n, a^n, b^n) \) is convergent. Moreover, \( u^n \) converges in L2-strong topology to the unique minimizer of \( J_\varepsilon \).

Conclusions

- General method for extracting velocity fields based on projected motion equation or optical flow equation

- Application to various flow visualization images

- High resolution (one vector at each pixel)

- Alternative to correlation-based PIV