Optimum Bifurcating-Tube Tree for Gas Transport

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Bronchial Tree as an Engineering Tree Model
Bifurcating Tube Tree Geometry

**Objective:**

To determine the distributions of the geometrical quantities as a function of the generation number.
Murray’s Law

Minimum Power Principle (Murray, CD 1926):

\[ d_n \propto 2^{-n/3} \]

Minimum Resistance (Rashevsky, N 1960)
Minimum Entropy (Wilson, TA 1967)
Minimum Volume (LaBarbera, M 1990)
Measurements of Diameter and Length

\[ d_n \propto 2^{-n/2} \]

\[ d_n \propto 2^{-2n/16} \]

\[ l_n \propto 2^{-n/4} \]
Underlying Physical Mechanisms

(1) Convection Zone

Minimum Power, Resistance and Entropy

(2) Diffusion Zone

Maximum Total Mass Diffusion Rate ?
Max-Min Problem

Cost Function: \[ P = \sum_{n=1}^{N} w_n x_n \]

Constraint: \[ \sum_{n=1}^{N} x_n^s = B \]

Solution: \[ (x_n)_{op} = B^{1/s} w_n^{1/(s-1)} \left( \sum_{n=1}^{N} w_n^{s/(s-1)} \right)^{-1/s} \]
\[ n = 1, 2, \ldots, N \]

For \( s > 1 \), Maximum of \( P \);

For \( s < 1 \), Minimum of \( P \).
Diameter Distribution and Maximum Gas Diffusion Rate

Diffusion Mass-Transfer Rate:

\[ m_n = D_{\text{diff}} \left( A_{\text{eff}} \right)_n \Delta C_n / h_n \]

\( (A_{\text{eff}})_n \) Effective diffusion area

\( \Delta C_n \) Change of gas concentration Across the wall
Relation between Mass Transfer Rate and Volume

The geometrical relations:

\[(A_{\text{eff}})_n \propto V_n^{\alpha D_2/3}\]

\[h_n \propto 2^{-3n/8} V_n^{1/2}\]

The cascade of mass transfer:

\[C_n \propto V_n^{-\gamma} \quad (n > N/2)\]

Diffusion mass-transfer rate:

\[\dot{m}_n = B_1 2^{3n/8} V_n^{1/s}\]

The parameter:

\[s = (\alpha D_2/3 - \gamma - 1/2)^{-1}\]
Maximizing Total Diffusion Mass-Transfer Rate:

\[ \dot{m}_T = \sum m_n = \sum w_n x_n \]

Constant Total Volume Constraint:

\[ \sum V_n = \sum x_n^s = B_2 \]

where the parameters are

\[ x_n = V_n^{1/s} \quad w_n = B_1 2^{3n/8} \]
At the optimal condition,

\[
(V_n)_{op}^{1/s} = B_2^{1/s} \sum w_n^{s/(s-1)} \left( \sum w_n^{s/(s-1)} \right)^{-1/s}
\]

Consistence condition requires

\[
s \rightarrow + \infty
\]

\[
(V_n)_{op} \propto 2^{3n/8}
\]
Diameter Distribution in Diffusion Zone

New law for the diffusion zone:

\[ d_n \propto 2^{-3n/16} \]

Murray’s law for the convection zone:

\[ d_n \propto 2^{-n/3} \]
Length Distribution, Minimum Weight and Structural Stability

**Minimizing total weight**

\[ W_T = \sum (W_T)_n \]

**Constant total surface constraint**

\[ \sum (W_T)_n^{D_{2/3}} = \text{const.} \]

**Structure stability requirement**

\[ l_n \propto W_n^{1/4} \]
Length Distribution for the Entire Tree

\[ l_n \propto 2^{-n/4} \]
Conclusions

Diameter distribution in the diffusion zone:

\[ d_n \propto 2^{-3n/16} \]

Length distribution:

\[ l_n \propto 2^{-n/4} \]

Bifurcation angle for min power & volume:

\[ \phi \approx 75^\circ \]

Bifurcation angle for min drag & surface:

\[ \phi \approx 102^\circ \]
Further Comments

- **Tree is not on a plane.**
  Determining rotational angle distribution: a non-trivial geometry problem

- **This work does not attempt to provide a teleological explanation for the physiology of bronchial trees.**